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Classical integrable defects as quasi Bäcklund transformations

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Abstract

We consider the algebraic setting of classical defects in discrete and continuous integrable theories. We derive the "equations of motion" on the defect point via the space-like and time-like description. We then exploit the structural similarity of these equations with the discrete and continuous Bäcklund transformations. And although these equations are similar they are not exactly the same to the Bäcklund transformations. We also consider specific examples of integrable models to demonstrate our construction, i.e. the Toda chain and the sine-Gordon model. The equations of the time (space) evolution of the defect (discontinuity) degrees of freedom for these models are explicitly derived.

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1. Introduction

The issue of quantum and classical integrable defects has been the subject of consistently increasing research interest in recent years [1–16]. Many results from various perspectives were produced for both quantum and classical integrable models and many interesting interconnections were pointed out. For instance the elucidation of the local defect as a "frozen" Bäcklund transformation [3,5] was a significant observation providing a novel way of understanding local defects for integrable classical field theories. And although the interpretation of the defect matrix as a"frozen" Bäcklund transformation is a notable piece of information, a systematic Hamilto-

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nian/algebraic description of this notion has been hitherto missing, and this is precisely the main objective in this paper.

More specifically, we shall introduce here the notion of *quasi Bäcklund* transformations for both discrete and continuous classical integrable models in the presence of local defects. This concept is essentially associated to the local equations of motion around the defect point. Note that in the usual Bäcklund transformation formulation one deals with a space and a time differential (difference) equation, and these two equations are simultaneously satisfied. In the presence of local defects however this is not the case, as the space and time differential (difference) equations that are obtained are not satisfied simultaneously. It is worth noting that these two equations in the frame of defects arise as local equations of motion around the defect point (more details on this will be presented later in the text, see also [14]). In the discrete case in particular one explicitly derives the time components of the Lax pair around the defects point, and these turn out to be slightly "deformed" compared to the bulk quantities. In the continuous case on the other hand due to imposed analyticity conditions around the defect point the time components of the Lax pair coincide with the left/right bulk theory quantities (for more details we refer the interested reader to [14]). These connections are of course remarkable and a detailed analysis is given in this article, however a deeper understanding of the particular solutions of these equations for various physically relevant models should be further pursued (see some relevant discussion in [12]).

Let us briefly outline the content on this paper: in the next section we briefly review the auxiliary linear problem for both discrete and continuous classical integrable models. We also recall the Darboux–Bäcklund transformations as suitable gauge transformations that leave the auxiliary linear problem invariant. The underlying classical algebras associated to the Lax pair are briefly described, and the notion of the "dual" formulation introduced in [17] is also presented. A note on the algebraic content of the Bäcklund transformations is also given based mainly on the fact that the auxiliary functions are versions of classical vertex operators [18]. In section 3 we focus on the problem under consideration i.e. the description of the local classical defects as quasi Bäcklund transformations. More precisely, we discuss the equations of motion associated to the defect degrees of freedom, and ascertain their resemblance to the t part of the Bäcklund transformation. We also consider defects along the t axis ("dual" description) and thus we are able to derive the x part of the Bäcklund transformation. Jump conditions on the fields and their time derivatives are also obtained similar to the ones along the x direction, and are part of the x Bäcklund transformation conditions. The underlying algebra for the Lax matrices as well as the time and space defect matrices are also discussed.

2. The general setting

We shall review in this section the main setting in describing continuous and discrete integrable classical models. The formulation we pursue here is based on two main building blocks, i.e. the auxiliary linear algebra on the one hand and the existence of an underlying classical algebra. We shall describe below both semi and fully discrete integrable models as well as continuous integrable theories. Moreover, we shall review the Darboux–Bäcklund transformations as suitable gauge transformation associated to the auxiliary linear problem.