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Analogies between random matrix ensembles and the one-component plasma in two-dimensions

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Abstract

The eigenvalue PDF for some well known classes of non-Hermitian random matrices — the complex Ginibre ensemble for example — can be interpreted as the Boltzmann factor for one-component plasma systems in two-dimensional domains. We address this theme in a systematic fashion, identifying the plasma system for the Ginibre ensemble of non-Hermitian Gaussian random matrices G, the spherical ensemble of the product of an inverse Ginibre matrix and a Ginibre matrix $G_1^{-1}G_2$, and the ensemble formed by truncating unitary matrices, as well as for products of such matrices. We do this when each has either real, complex or real quaternion elements. One consequence of this analogy is that the leading form of the eigenvalue density follows as a corollary. Another is that the eigenvalue correlations must obey sum rules known to characterise the plasma system, and this leads us to an exhibit of an integral identity satisfied by the two-particle correlation for real quaternion matrices in the neighbourhood of the real axis. Further random matrix ensembles investigated from this viewpoint are self dual non-Hermitian matrices, in which a previous study has related to the one-component plasma system in a disk at inverse temperature $\beta = 4$, and the ensemble formed by the single row and column of quaternion elements from a member of the circular symplectic ensemble.

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1. Introduction

In random matrix theory there are a number of distinguished ensembles — the classical cases — for which the eigenvalue probability density function (PDF) can be calculated explicitly. For example, the classical Gaussian orthogonal ensemble consisting of real symmetric matrices $\frac{1}{2}(X + X^T)$, where X is an $N \times N$ standard real Gaussian matrix, has its joint eigenvalue PDF proportional to

$$\prod_{l=1}^{N} e^{-\frac{1}{2}\lambda_l^2} \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|. \tag{1.1}$$

This explicit expression was known to Wigner (see [63] and references therein).

Wigner [72] (reprinted in [63]) also observed that (1.1), upon being written in the form $e^{-\beta U}$, is identical to the Boltzmann factor for the classical gas in one-dimension with total potential energy

$$U = \frac{1}{2}\sum_{l=1}^{N} \lambda_l^2 - \sum_{1 \le j < k \le N} \log|\lambda_k - \lambda_j|, \qquad \lambda_l \in \mathbb{R}, \tag{1.2}$$

interacting at inverse temperature $\beta = 1$. The first term in (1.2) represents an harmonic attraction towards the origin, and the second is a pairwise logarithmic repulsion between the particles in the gas. The pair potential

$$-\log|z - w|, \tag{1.3}$$

with $z, w \in \mathbb{C}$, is well known as the solution of the two-dimensional Poisson equation with free boundary conditions, and thus the pair interaction in (1.2) is that experienced by N two-dimensional unit charges confined to a line. To understand the origin of the one-body term $\frac{1}{2}\lambda^2$ from this perspective, suppose there is a smeared out neutralising background charge density $-\sigma(\lambda)$, supported on the interval $[-L, L]$. The one-body term must be the electrostatic energy created by this background charge, and thus we must have

$$\frac{1}{2}\lambda^2 + C = \int_{-L}^{L} \sigma(y)\log|\lambda - y|\, dy, \qquad \lambda \in [-L, L], \tag{1.4}$$

for some constant C independent of λ. The solution of this integral equation, with the requirement that $\sigma(y)$ vanishes at $y = \pm L$ is (see e.g. [25, Prop. 1.4.3])

$$\sigma(y) = \frac{L}{\pi}\sqrt{1 - (y/L)^2}. \tag{1.5}$$

Charge neutrality requires $\int_{-L}^{L} \sigma(y)\, dy = N$, which in turn implies

$$L = \sqrt{2N}. \tag{1.6}$$

In random matrix theory we recognise (1.5) and (1.6) as the Wigner semi-circle law for the eigenvalue density of large random real symmetric matrices; see e.g. [60]. This in fact is one of the derivations of the law given by Wigner himself [72].

Our interest in this paper is in analogies between eigenvalue PDFs with two-dimensional support in the complex plane, and the Boltzmann factor for one-component log-potential classical