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Domain wall equations, Hessian of superpotential, and Bogomol'nyi bounds

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Abstract

An important question concerning the classical solutions of the equations of motion arising in quantum field theories at the BPS critical coupling is whether all finite-energy solutions are necessarily BPS. In this paper we present a study of this basic question in the context of the domain wall equations whose potential is induced from a superpotential so that the ground states are the critical points of the superpotential. We prove that the definiteness of the Hessian of the superpotential suffices to ensure that all finite-energy domain-wall solutions are BPS. We give several examples to show that such a BPS property may fail such that non-BPS solutions exist when the Hessian of the superpotential is indefinite.

1. Introduction

Domain walls, vortices, monopoles, and instantons are classical solutions of various equations of motion in quantum field theory describing particle-like behavior in interaction dynamics in one, two, three, and four spatial dimensions, respectively [50]. Due to the complexity of these equations it is difficult to obtain a full understanding of their solutions in general settings. Fortunately, at certain critical coupling limits, enormous insight into the solutions may be obtained

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from investigating the so-called BPS solutions, after the pioneering works of Bogomol'nyi [12] and Prasad and Sommerfield [38]. Mathematically, at such critical limits, often referred to as the BPS limits, the original second-order equations of motion permit an elegant reduction into some first-order equations, called the BPS equations, whose solutions are automatically the energy minimizers of the models, and hence, the most physically relevant field configurations. The minimum energy values achieved by the BPS solutions are called the BPS or Bogomol'nyi bounds and are often expressed in terms of the topological charges of the models. Physically, the BPS limits correspond to the situations that the Higgs scalar and gauge boson masses are equal in the Abelian Higgs model [26], the Higgs doublet and Z vector boson masses are equal in the electroweak theory [5], both for vortices, the Higgs potential vanishes but symmetry is spontaneously broken at an arbitrary level for monopoles [28], and only gauge fields are present for instantons [2,39,34]. There has also been some study on why BPS bounds exist in quantum field theory [27]. For more recent developments of the application of the BPS reduction in supersymmetric field theory, see [30,44,45,49] for surveys.

At the BPS coupling then an important question arises: Are the original second-order equations of motion equivalent to their BPS-reduced first-order equations? In other words, are all finite-energy critical points of the field-theoretical energy functional the solutions of the BPS equations, and hence, attain the BPS bounds? For the Abelian Higgs vortices, Taubes proved [28, 47] that the answer is yes, and for the non-Abelian Yang–Mills–Higgs monopoles, he established [48] that the answer is no such that there are nonminimal solutions of the Yang–Mills–Higgs equations in the BPS coupling which are not solutions to the BPS equations. For the Yang–Mills instantons, Sibner, Sibner, and Uhlenbeck [46] gave a no answer to the question and proved the existence of a nonminimal solution when the instanton charge is zero, and Bor [14], Parker [35], and Sadun and Segert [40–42] obtained the existence of nonminimal solutions when the instanton charge is non-zero and non-unit, by exploring an equivariant structure in the geometric construction of the Yang–Mills fields. Much earlier, inspired by the work of Taubes [48] based on a Morse theory consideration, Manton [32] and Klinkhamer and Manton [29] investigated the possible existence of saddle point solutions, also referred to as sphalerons whose energy gives rise to the height of a barrier for tunneling between two vacuum states of distinct topological charges, in the Weinberg–Salam electroweak theory, by a demonstration that the field configuration space of the bosonic sector possesses a non-contractible loop. In [24] Forgács and Horváth presented a series of examples of field-theoretical models in one, two, and three spatial dimensions that allow non-contractible loops in their configuration spaces. Hence these models may be candidates for the occurrence of saddle-point solutions and host barriers to topological vacuum tunneling as in the electroweak theory [29,32]. These studies prompt a systematic investigation of the existence of non-BPS solutions at the BPS limits, referred to here as the BPS problem.

Due to the complicated structures of various quantum field theory models, it will be difficult to obtain a thorough understanding of the BPS problem in its fully general setting. In this paper, as a starting point, we consider a general domain wall model where the field configuration is an n-component real scalar field. As the name suggests, a domain wall is a dimensionally reduced field configuration that is embedded into, and links two distinct domain phases realized as ground states of, the full field-theoretical model. Well-known classical examples of domain walls include the solutions of the sine–Gordon equations describing a domain transition in terms of the magnetization orientation angle in the Landau–Lifshitz theory of magnetism and the solutions of the Ginzburg–Landau equations connecting the normal and superconducting phases so that the sign of its energy, called the surface energy, classifies superconductivity. In modern physics, domain walls find applications in a wide range of subjects such as phase transitions in supersym-