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Functional relations from the Yang–Baxter algebra: Eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions

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Abstract

In this work we consider a functional method in the theory of exactly solvable models based on the Yang–Baxter algebra. Using this method we derive the eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions for general values of the anisotropy and boundary parameters.

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1. Introduction

Functional equations methods appeared in the theory of exactly solvable lattice models intimately connected with Baxter's commuting transfer matrix method [1]. In the early seventies Baxter introduced in his pioneer work [1] the concept of Q-operators and T–Q relations determining the eigenvalues of the transfer matrix of the corresponding vertex model.

The complex calculations involved in Baxter's construction of Q-operators seems to have restricted its use and other functional methods, such as the Reshetikhin's analytical Bethe ansatz, were employed instead to obtain the spectrum of transfer matrices related with quantum Kac–

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Moody algebras [2]. However we remark here the recent progresses in the construction of Q-operators by employing quantum algebras representation theoretic methods [3,4].

On the other hand, the advent of the algebraic Bethe ansatz in the late seventies provided a systematic approach to find the eigenvalues and eigenvectors of transfer matrices of integrable vertex models. This method, originally proposed by Takhtadzhan and Faddeev [5], is based on the existence of a pseudovacuum or reference state and appropriate commutation rules arising from the Yang–Baxter algebra, which is a common algebraic structure associated with integrable vertex models.

Although the algebraic Bethe ansatz method is a powerful tool exhibiting a rich mathematical structure, its implementation for models that do not possess a trivial reference state is still an obstacle to be overcomed. The aim of this paper is to show that the Yang–Baxter algebra can be explored in order to generate functional relations determining the spectrum of transfer matrices. In order to illustrate that we consider the XXZ model with non-diagonal twists and open boundaries for general values of the anisotropy and boundary parameters. Besides the relevance of studying models with general boundary conditions in the context of statistical mechanics [6], the XXZ model with non-diagonal twists and open boundaries are both included in a class of models where a trivial reference state is absent.

This paper is organized as follows. In Section 2 we describe the XXZ model with general toroidal boundary conditions. In particular, we discuss the case of non-diagonal twists and we derive functional relations for its eigenvalues making use of the Yang–Baxter algebra. In Section 3 we approach the eigenvalue problem for the XXZ model with non-diagonal open boundaries using the algebraic-functional method devised in Section 2. Concluding remarks are discussed in Section 4 and in Appendices A–D we give some extra results and technical details.

2. The XXZ model with non-diagonal twisted boundary conditions

The advent of the quantum inverse scattering method [5,7] was an important stage in the development of the theory of exactly solvable quantum systems. This method unveiled a deep connection between solutions of the Yang–Baxter equation, quantum integrable systems and exactly solvable lattice models of statistical mechanics in two dimensions [8]. In statistical mechanics an important role is played by vertex models whose respective transfer matrix is constructed from local Boltzmann weights contained in an operator $\mathcal{L}_{\mathcal{A}j}$. Let V be a finite dimensional linear space, the integrability of the vertex model is achieved when the operator valued function $\mathcal{L}: \mathbf{C} \rightarrow \mathrm{End}(V \otimes V)$ is a solution of the Yang–Baxter equation, namely

$$\mathcal{L}_{12}(\lambda - \mu)\mathcal{L}_{13}(\lambda)\mathcal{L}_{23}(\mu) = \mathcal{L}_{23}(\mu)\mathcal{L}_{13}(\lambda)\mathcal{L}_{12}(\lambda - \mu), \tag{1}$$

defined in the space $V_1 \otimes V_2 \otimes V_3$. Here we use the standard notation $\mathcal{L}_{ij} \in \mathrm{End}(V_i \otimes V_j)$. The complex valued operator $\mathcal{L}_{\mathcal{A}j}(\lambda)$ can be viewed as a matrix in the space of states $\mathcal{A}$ denoting for instance the horizontal degrees of freedom of a square lattice, while its matrix elements are operators acting non-trivially in the jth position of $\bigotimes_{i=1}^{L} V_i$. In its turn the space V_j represents the space of states of the vertical degrees of freedom at the jth site of a chain of lengh L.

The transfer matrix of the corresponding vertex model can be written in terms of the monodromy matrix $\mathcal{T}_{\mathcal{A}}(\lambda)$ defined by the following ordered product

$$\mathcal{T}_{\mathcal{A}}(\lambda) = \mathcal{L}_{\mathcal{A}L}(\lambda)\mathcal{L}_{\mathcal{A}L-1}(\lambda) \cdots \mathcal{L}_{\mathcal{A}1}(\lambda). \tag{2}$$