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Complete normal ordering 1: Foundations

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Abstract

We introduce a new prescription for quantising scalar field theories (in generic spacetime dimension and background) perturbatively around a true minimum of the full quantum effective action, which is to 'complete normal order' the bare action of interest. When the true vacuum of the theory is located at zero field value, the key property of this prescription is the automatic cancellation, to any finite order in perturbation theory, of all tadpole and, more generally, all 'cephalopod' Feynman diagrams. The latter are connected diagrams that can be disconnected into two pieces by cutting one internal vertex, with either one or both pieces free from external lines. In addition, this procedure of 'complete normal ordering' (which is an extension of the standard field theory definition of normal ordering) reduces by a substantial factor the number of Feynman diagrams to be calculated at any given loop order. We illustrate explicitly the complete normal ordering procedure and the cancellation of cephalopod diagrams in scalar field theories with non-derivative interactions, and by using a point splitting 'trick' we extend this result to theories with derivative interactions, such as those appearing as non-linear σ-models in the world-sheet formulation of string theory. We focus here on theories with trivial vacua, generalising the discussion to non-trivial vacua in a follow-up paper.

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1. Overview

One of the important tools in quantum field theory is the procedure of 'normal ordering' [1], $\mathcal{O}(\phi) \to :\mathcal{O}(\phi):$, that is invaluable, e.g., when defining products of field operators at coincident points and when evaluating correlation functions using Wick's theorem. A crucial property is that expectation values of normal-ordered operators vanish in the *free* theory: $\langle :\mathcal{O}(\phi): \rangle_0 = 0$. There are various formulations of this notion [2], including creation–annihilation operator normal ordering, conformal normal ordering, functional integral normal ordering, etc., which are often interrelated. The following is one concise definition of standard normal ordering,

$$:\mathcal{O}(\phi): = \mathcal{O}(\delta_X)\, e^{-\frac{1}{2}\int_z\int_w \mathcal{G}(z,w)X(z)X(w)+\int X\phi}\Big|_{X=0}, \tag{1.1}$$

where $\mathcal{G}(z, w)$ is the *free* Feynman propagator of the theory, and δ_X is a functional derivative with respect to the (unphysical) source X. For example, for an interaction term $\mathcal{O}(\phi) = \frac{1}{N!} g_N \phi^N$ in the action of interest, upon normal ordering we obtain:

$$:\phi^N: = B_N(\phi, -\mathcal{G}, 0, \dots, 0), \tag{1.2}$$

with $B_N(a_1, \dots, a_N)$ a complete Bell polynomial[1] [3,4] and $\mathcal{G}$ the Feynman propagator at coincident points. In the interaction picture of quantum field theory this standard normal ordering of the action produces counterterms that eradicate Feynman diagrams with internal lines that begin and end on the same internal vertex [5], such as the one-loop two-point amplitude in ϕ^4, as well as some (but not all) tadpole diagrams in ϕ^3 scalar field theory.[2] These counterterms, call them $\delta_{::}g_n$, are read off from the right-hand side of (1.2), and in particular $\frac{1}{N!} g_N :\phi^N:$ equals $\frac{1}{N!} g_N \phi^N$ plus a polynomial $\sum_{n=0}^{N-1} \frac{1}{n!}\delta_{::}g_n \phi^n$ with:

$$\delta_{::}g_n = \frac{g_N}{(N-n)!} B_{N-n}(0, -\mathcal{G}, 0, \dots, 0). \tag{1.4}$$

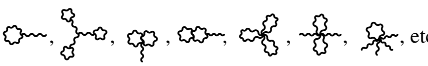
This quick exercise yields one of the terms contributing to $\delta_{::}g_n$, and there may be various such contributions if we also sum over N (as we typically should to preserve renormalisability). Substituting these counterterms into the generating function of interest is clearly equivalent to having started from a path integral with a normal-ordered action in the first place.

In 2 dimensions and in the absence of derivative interactions these counterterms make all correlation functions (of elementary fields) UV-finite. But this still does not guarantee that we are quantising the theory around a true minimum of the full quantum effective action, given that there will generically be an infinite number of (physical and unphysical) tadpole diagrams. In addition, there will generically also be an infinite number of *cephalopod* diagrams in the resulting amplitudes, and these can also be removed by an appropriate choice of counterterms.

Cephalopods [6] are generalisations of the familiar 'tadpole' diagrams: the 1PI version of a 'cephalopod' diagram has an arbitrary number $(0, 1, 2, ..)$ of external legs and an arbitrary

1 These satisfy $B_n(a_1, \dots, a_n) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}(0, a_2, \dots, a_{n-r}) a_1^r$ and $B_n(a_1, \dots, a_n) z^n = B_n(z a_1, \dots, z^n a_n)$ and are defined by the generating function:

$$\sum_{n=0}^{\infty} \frac{1}{n!} B_n(a_1, \dots, a_n) z^n := \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n!} a_n z^n\Big). \tag{1.3}$$

2 Other diagrams that are cancelled by normal ordering are: , etc.