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Center of the universal Askey–Wilson algebra at roots of unity

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Abstract

Inspired by a profound observation on the Racah–Wigner coefficients of $U_q(\mathfrak{sl}_2)$, the Askey–Wilson algebras were introduced in the early 1990s. A universal analog $\triangle_q$ of the Askey–Wilson algebras was recently studied. For q not a root of unity, it is known that $Z(\triangle_q)$ is isomorphic to the polynomial ring of four variables. A presentation for $Z(\triangle_q)$ at q a root of unity is displayed in this paper. As an application, a presentation for the center of the double affine Hecke algebra of type $(C_1^\vee, C_1)$ at roots of unity is obtained.

1. Introduction

Throughout this paper an algebra $\mathcal{A}$ is meant to be an associative algebra with unit and let $Z(\mathcal{A})$ denote the center of an algebra $\mathcal{A}$.

Fix a complex scalar $q \neq 0$. In [36] Zhedanov proposed the Askey–Wilson algebras which involve five extra parameters $\varrho, \varrho^*, \eta, \eta^*, \omega$. Given these scalars the *Askey–Wilson algebra* is an algebra over the complex number field $\mathbb{C}$ generated by K_0, K_1, K_2 subject to the relations

$$qK_1K_2 - q^{-1}K_2K_1 = \omega K_1 + \varrho K_0 + \eta^*,$$

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$$qK_2K_0 - q^{-1}K_0K_2 = \omega K_0 + \varrho^* K_1 + \eta,$$
$$qK_0K_1 - q^{-1}K_1K_0 = K_2.$$

These algebras are named after R. Askey and J. Wilson since the algebras can also describe a hidden relation between the three-term recurrence relation and the q-difference equation of Askey–Wilson polynomials [1]. Under the mild assumptions $q^4 \neq 1$, $\varrho \neq 0$ and $\varrho \neq 0$ substitute

$$K_0 = -\frac{\sqrt{\varrho^*}A}{q^2 - q^{-2}}, \qquad K_1 = -\frac{\sqrt{\varrho}B}{q^2 - q^{-2}}, \qquad K_2 = \frac{\omega}{q - q^{-1}} - \frac{\sqrt{\varrho\varrho^*}C}{q^2 - q^{-2}}$$

into the defining relations of the Askey–Wilson algebra. The resulting relations become that each of

$$A + \frac{qBC - q^{-1}CB}{q^2 - q^{-2}}, \qquad B + \frac{qCA - q^{-1}AC}{q^2 - q^{-2}}, \qquad C + \frac{qAB - q^{-1}BA}{q^2 - q^{-2}} \tag{1}$$

is equal to a scalar. By interpreting the elements in (1) as central elements, it turns into the so-called *universal Askey–Wilson algebra* $\triangle_q$ [33]. Let us denote $\triangle = \triangle_q$ for brevity.

Let α, β, γ denote the central elements of $\triangle$ obtained from multiplying the elements (1) by $q + q^{-1}$, respectively. Motivated by Zhedanov [36, §1], the distinguished central element

$$qABC + q^2A^2 + q^{-2}B^2 + q^2C^2 - qA\alpha - q^{-1}B\beta - qC\gamma \tag{2}$$

is called the *Casimir element* of $\triangle$. For q not a root of unity, the center of $\triangle$ has been shown in [33, Theorem 8.2] to be the four-variable polynomial ring over $\mathbb{C}$ generated by α, β, γ and the Casimir element (2). The inspiration of our study on $Z(\triangle)$ at roots of unity comes from the quantum group $U_q'(\mathfrak{so}_3)$. The quantum group $U_q'(\mathfrak{so}_n)$ [9] is not Drinfeld–Jimbo type but plays the important roles in the study of q-Laplace operators and q-harmonic polynomials [17, 27], q-ultraspherical polynomials [31], quantum homogeneous spaces [26], nuclear spectroscopy [10], $(2+1)$-dimensional quantum gravity [25,24] and so on. For $n = 3$ the quantum group is exactly the Askey–Wilson algebra with $q^4 \neq 1$, $\varrho = 1$, $\varrho^* = 1$, $\eta = 0$, $\eta^* = 0$, $\omega = 0$. According to [27, §4] the Casimir element of $U_q'(\mathfrak{so}_3)$ is defined to be

$$q(q^2 - q^{-2})K_0K_1K_2 - q^2K_0^2 - q^{-2}K_1^2 - q^2K_2^2. \tag{3}$$

As far as we know, Odesskii [29, Theorem 4] first found three additional central elements of $U_q'(\mathfrak{so}_3)$ at roots of unity defined as follows. Assume that q is a primitive dth root of unity and set

$$đ = \begin{cases} d & \text{if } d \text{ is odd,} \\ d/2 & \text{if } d \text{ is even.} \end{cases}$$

Denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set of the nonnegative integers. For each $n \in \mathbb{N}$ define

$$T_n(X) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \left(\binom{n-i}{i} + \binom{n-i-1}{i-1} \right) X^{n-2i}. \tag{4}$$

Here $\binom{n}{-1}$ for $n \in \mathbb{N}$ and $\binom{-1}{-1}$ are interpreted as 0 and 1, respectively. Note that $\frac{1}{2}T_n(2X)$ is the Chebyshev polynomial of the first kind. Then

$$\Gamma_i = T_{đ}(-(q^2 - q^{-2})K_i) \qquad \text{for all } i \in \mathbb{Z}/3\mathbb{Z}$$

are central in $U_q'(\mathfrak{so}_3)$. A proof can be found in [11, Lemma 2].