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Algebraic Bethe ansatz for the quantum group invariant open XXZ chain at roots of unity

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Abstract

For generic values of q, all the eigenvectors of the transfer matrix of the $U_q sl(2)$-invariant open spin-$1/2$ XXZ chain with finite length N can be constructed using the algebraic Bethe ansatz (ABA) formalism of Sklyanin. However, when q is a root of unity ($q = e^{i\pi/p}$ with integer $p \geq 2$), the Bethe equations acquire continuous solutions, and the transfer matrix develops Jordan cells. Hence, there appear eigenvectors of two new types: eigenvectors corresponding to continuous solutions (exact complete p-strings), and generalized eigenvectors. We propose general ABA constructions for these two new types of eigenvectors. We present many explicit examples, and we construct complete sets of (generalized) eigenvectors for various values of p and N.

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1. Introduction

In the pantheon of anisotropic integrable quantum spin chains, one model stands out for its high degree of symmetry: the $U_q sl(2)$-invariant open spin-$1/2$ XXZ quantum spin chain, whose Hamiltonian is given by [1]

$$H = \sum_{k=1}^{N-1} \left[\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \tfrac{1}{2}(q+q^{-1}) \sigma_k^z \sigma_{k+1}^z \right] - \tfrac{1}{2}(q-q^{-1}) \left(\sigma_1^z - \sigma_N^z \right) , \tag{1.1}$$

where N is the length of the chain, $\vec{\sigma}$ are the usual Pauli spin matrices, and $q = e^{\eta}$ is an arbitrary complex parameter. As is true for generic quantum integrable models, the Hamiltonian is a member of a family of commuting operators that can be obtained from a transfer matrix [2]; and the eigenvalues of the transfer matrix can be obtained in terms of admissible solutions $\{\lambda_k\}$ of the corresponding set of Bethe equations [3,2,1][1]

$$\begin{aligned}
&\mathrm{sh}^{2N}\left(\lambda_k + \tfrac{\eta}{2}\right) \prod_{\substack{j \neq k \\ j=1}}^{M} \mathrm{sh}(\lambda_k - \lambda_j - \eta)\, \mathrm{sh}(\lambda_k + \lambda_j - \eta) \\
&\quad = \mathrm{sh}^{2N}\left(\lambda_k - \tfrac{\eta}{2}\right) \prod_{\substack{j \neq k \\ j=1}}^{M} \mathrm{sh}(\lambda_k - \lambda_j + \eta)\, \mathrm{sh}(\lambda_k + \lambda_j + \eta) , \\
&\qquad k = 1, 2, \ldots, M , \qquad M = 0, 1, \ldots, \lfloor \tfrac{N}{2} \rfloor ,
\end{aligned} \tag{1.2}$$

where $\lfloor k \rfloor$ denotes the integer not greater than k.

When the anisotropy parameter η takes the values $\eta = i\pi/p$ with integer $p \geq 2$, and therefore $q = e^{\eta}$ is a root of unity, several interesting new features appear. In particular, the symmetry of the model is enhanced (for example, an $sl(2)$ symmetry arises from the so-called divided powers of the quantum group generators); the Hamiltonian has Jordan cells [4–6]; and the Bethe equations (1.2) admit continuous solutions [7], in addition to the usual discrete solutions (the latter phenomenon also occurs for the closed XXZ chain [8–12]).

We have recently found [7] significant numerical evidence that the Bethe equations have precisely the right number of admissible solutions to describe all the distinct (generalized) eigenvalues of the model's transfer matrix, even at roots of unity.

We focus here on the related problem of constructing, via the algebraic Bethe ansatz, all 2^N (generalized) eigenvectors of the transfer matrix. For generic q, the construction of these eigenvectors is similar to the one for the simpler spin-$1/2$ XXX chain: to each admissible solution of the Bethe equations, there corresponds a Bethe vector, which is a highest-weight state of $U_q sl(2)$ [1,13,14]; and lower-weight states can be obtained by acting on the Bethe vector with the quantum-group lowering operator F.

However, at roots of unity $q = e^{i\pi/p}$ with integer $p \geq 2$, we find that there are two additional features:

i. Certain eigenvectors must be constructed using the continuous solutions noted above. These solutions contain p equally-spaced roots (so-called exact complete p-strings), whose centers

[1] In order to reduce the size of formulas, we denote the hyperbolic sine function (sinh) by sh.