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The analytic renormalization group

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Abstract

Finite temperature Euclidean two-point functions in quantum mechanics or quantum field theory are characterized by a discrete set of Fourier coefficients G_k, $k \in \mathbb{Z}$, associated with the Matsubara frequencies $\nu_k = 2\pi k/\beta$. We show that analyticity implies that the coefficients G_k must satisfy an infinite number of model-independent linear equations that we write down explicitly. In particular, we construct "Analytic Renormalization Group" linear maps A_μ which, for any choice of cut-off μ, allow to express the low energy Fourier coefficients for $|\nu_k| < \mu$ (with the possible exception of the zero mode G_0), together with the real-time correlators and spectral functions, in terms of the high energy Fourier coefficients for $|\nu_k| \ge \mu$. Operating a simple numerical algorithm, we show that the exact universal linear constraints on G_k can be used to systematically improve any random approximate data set obtained, for example, from Monte-Carlo simulations. Our results are illustrated on several explicit examples.

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1. General presentation

Consider the space $\mathscr{M}$ of arbitrary two-point functions between bosonic operators A and B in Quantum Mechanics or Quantum Field Theory, at finite temperature $T = 1/\beta$.[1] As is well-known and will be reviewed in details in Section 2, the space $\mathscr{M}$ can be presented in several equivalent ways. One can consider various real-time two-point functions (advanced, retarded, time-ordered, etc.), which turn out to be all related to each other, since their Fourier transforms can be expressed in terms of a unique spectral function $\rho(\omega)$. Alternatively, one can work with the Euclidean-time two-point function $G(\tau)$. By the KMS condition, G is periodic and can be expanded in Fourier series,

$$G(\tau) = \frac{1}{\beta}\sum_{k\in\mathbb{Z}} G_k e^{-i\nu_k\tau}\,, \tag{1.1}$$

where the Matsubara frequencies are defined by

$$\nu_k = 2\pi k/\beta\,. \tag{1.2}$$

We shall often refer to the set of Fourier coefficients $(G_k)_{k\in\mathbb{Z}}$ as the "data" which encodes the two-point function. In a generic strongly coupled quantum mechanical model, this data can only be computed numerically, using Monte-Carlo numerical simulations. Analytic non-perturbative methods exist only in rare occasions.[2]

By Carlson's theorem [2], the real-time and Euclidean-time points of view are equivalent: the continuous spectral function $\rho(\omega)$ can be expressed in terms of the discrete set of Fourier coefficients G_k and vice-versa, under some very general assumptions that are valid in all known interesting physical theories.[3] The map between the real-time and the Euclidean-time formalism is quite interesting and will be discussed very explicitly below.

The two-point functions must satisfy general well-known constraints that follow straightforwardly from the definitions and the spectral decomposition, see Section 2. For example, on top of being β-periodic, $G(\tau)$ is analytic except at the points $\tau = k\beta$, $k \in \mathbb{Z}$, where it is discontinuous if A and B do not commute. This implies in particular that $G_k = O(1/k)$ at large $|k|$. The Fourier coefficients also satisfy reality and positivity constraints depending on the reality properties of A and B. We shall call $\mathscr{F}$ the real vector space of β-periodic functions satisfying all these standard model-independent constraints.

One of the main goal of the present work is to show that $\mathscr{M}$ is a linear subspace of $\mathscr{F}$ of infinite codimension. This may come as a surprise. It means that a typical set of Fourier coefficients $(G_k)_{k\in\mathbb{Z}}$ satisfying all the usual constraints is actually inconsistent! Our central result is to show that *the Fourier coefficients must always obey an infinite set of universal, model-independent, linear equations*. For reasons that will become clear below, we call these equations "Analytic Renormalization Group" (ARG) equations. We shall write down these equations very explicitly in Section 3 and use them extensively in Sections 4 and 5.

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[1] We focus on the case of bosonic operators in the present paper. The case of fermionic operators can be discussed along the same lines, with minor and straightforward modifications.

[2] See e.g. [1] for a recent example from which the investigations presented in this paper originated.

[3] In Quantum Field Theory, two-point functions of local operators do not in general satisfy the hypothesis of Carlson's theorem, due to the usual UV divergences at coinciding points. These divergences are governed by the Operator Product Expansion which, in asymptotically free theories, can be reliably computed in perturbation theory. This problem is handled in a standard way: one either considers smeared versions of the local operators or, more generally, one subtracts explicitly the diverging piece in the correlator using the OPE.