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A note on generalized hypergeometric functions, KZ solutions, and gluon amplitudes

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Abstract

Some aspects of Aomoto's generalized hypergeometric functions on Grassmannian spaces $Gr(k+1, n+1)$ are reviewed. Particularly, their integral representations in terms of twisted homology and cohomology are clarified with an example of the $Gr(2,4)$ case which corresponds to Gauss' hypergeometric functions. The cases of $Gr(2,n+1)$ in general lead to $(n+1)$-point solutions of the Knizhnik–Zamolodchikov (KZ) equation. We further analyze the Schechtman–Varchenko integral representations of the KZ solutions in relation to the $Gr(k+1,n+1)$ cases. We show that holonomy operators of the so-called KZ connections can be interpreted as hypergeometric-type integrals. This result leads to an improved description of a recently proposed holonomy formalism for gluon amplitudes. We also present a (co)homology interpretation of Grassmannian formulations for scattering amplitudes in $\mathcal{N}=4$ super Yang–Mills theory.

1. Introduction

Recently, much attention is paid to Grassmannian formulations of scattering amplitudes in $\mathcal{N}=4$ super Yang–Mills theory. The Grassmannian formulations are initially proposed (see, *e.g.*, [1–6]) to make it manifest that the $\mathcal{N}=4$ super Yang–Mills amplitudes are invariant under the dual superconformal symmetry [7] at tree level. For more recent developments, see, *e.g.*, [8–13].

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As discussed in [12,13], these Grassmannian formulations have revived interests in a purely mathematical subject, *i.e.*, generalized hypergeometric functions on Grassmannian spaces $Gr(k+1, n+1)$, which were introduced and developed by Gelfand [14] and independently by Aomoto [15] many years ago. In relation to physics, it has been known that solutions of the Knizhnik–Zamolodchikov (KZ) equation in conformal field theory are expressed in terms of the generalized hypergeometric functions [16–20]. One of the main goals of this note is to present a clear and systematic review on these particular topics in mathematical physics. Particularly, we revisit integral representations of the KZ solutions by Schechtman and Varchenko [17,18] and analyze them in terms of a bilinear construction of hypergeometric integrals, using twisted homology and cohomology. Along the way, we also consider in detail Gauss' original hypergeometric functions in Aomoto's framework so as to familiarize ourselves to the concept of twisted homology and cohomology.

Another goal of this note is to study and understand analytic aspects of the holonomy operator of the so-called KZ connection. The holonomy of the KZ connection is first introduced by Kohno [21] (see also Appendix 4 in [15]) as a monodromy representation of the KZ equation in a form of the iterated integral [22]. Inspired by Kohno's result and Nair's observation [23] on the maximally helicity violating (MHV) amplitudes of gluons (also called the Parke–Taylor amplitudes [24]) in supertwistor space, the author has recently proposed a novel framework of deriving gluon amplitudes [25] where an S-matrix functional for the gluon amplitudes is defined in terms of the holonomy operator of a certain KZ connection. This framework, what we call the holonomy formalism, is intimately related to braid groups and Yangian symmetries. As mentioned in [26], the holonomy formalism also suggests a natural origin of the dual conformal symmetries. Towards the end of this note we would provide more rigorous mathematical foundations of the holonomy formalism and present an improved description of it. Lastly, we also consider the more familiar Grassmannian formulations of gluon amplitudes in the same framework. Namely, we analyze integral representations of the Grassmannian formulations and present a (co)homology interpretation of those integrals.

This note is organized as follows. In the next section we review some formal results of Aomoto's generalized hypergeometric functions on $Gr(k+1, n+1)$, based on textbooks by Japanese mathematicians [15,27–29]. We present a review in a pedagogical fashion since these results are not familiar enough to many physicists. In section 3 we consider a particular case $Gr(2, n+1)$ and present its general formulation. In section 4 we further study the case of $Gr(2, 4)$ which reduces to Gauss' hypergeometric function. Imposing permutation invariance among branch points, we here obtain new realizations of the hypergeometric differential equation in a form of a first order Fuchsian differential equation.

In section 5 we apply Aomoto's results to the KZ equation. We first focus on four-point KZ solutions and obtain them in a form of the hypergeometric integral. We then show that $(n+1)$-point KZ solutions in general can be represented by generalized hypergeometric functions on $Gr(2, n+1)$. We further consider the Schechtman–Varchenko integral representations of the KZ solutions in this context. The $(n+1)$-point KZ solutions can also be represented by the hypergeometric-type integrals on $Gr(k+1, n+1)$ but we find that there exist ambiguities in the construction of such integrals for $k \geq 2$. In section 6 we review the construction of the holonomy operators of the KZ connections. We make a (co)homology interpretation of the holonomy operator and obtain a better understanding of analytic properties of the holonomy operator.

The holonomy operator gives a monodromy representation of the KZ equation, which turns out to be a linear representation of a braid group. This mathematical fact has been one of the essential ingredients in the holonomy formalism for gluon amplitudes. In section 7 we briefly