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Differential equations and dispersion relations for Feynman amplitudes. The two-loop massive sunrise and the kite integral

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Abstract

It is shown that the study of the imaginary part and of the corresponding dispersion relations of Feynman graph amplitudes within the differential equations method can provide a powerful tool for the solution of the equations, especially in the massive case.

The main features of the approach are illustrated by discussing the simple cases of the 1-loop self-mass and of a particular vertex amplitude, and then used for the evaluation of the two-loop massive sunrise and the QED kite graph (the problem studied by Sabry in 1962), up to first order in the $(d-4)$ expansion.

1. Introduction

In the last years we have assisted to an impressive increase in our knowledge of the mathematical structures that appear in multiloop Feynman integrals, thanks to the combined use of various computational techniques, such as to the method of differential equations [1–3], the introduction of a class of special functions (dubbed originally harmonic polylogarithms, HPLs [4,5], they

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came out to be a subset of the much larger class of multiple polylogarithms, MPLs, see [6–9] and references therein), the definition of a so-called canonical basis [10] for dealing with increasingly larger systems of differential equations and the use of the Magnus exponentiation [11].

However, most of the above results have been obtained in the massless limit; indeed, the situation for massive amplitudes is different, as the two-loop massive sunrise (which has three propagators only) is still the object of thorough investigation [12–20]. A general approach to the study of arbitrarily complicated systems of differential equations within difference field theory has been recently proposed in [21].

In this paper we will show that the study of the imaginary parts and related dispersion relations satisfied by the Feynman amplitudes, within the differential equation frame, can provide another useful practical tool for their evaluation in the massive case as well.

The imaginary parts of Feynman graphs can be obtained in various ways. To start with, one can use Cutkosky–Veltman rule [22–24] for integrating directly the loop momenta in the very definition of the graphs. When the d-continuous dimensional regularization is used, nevertheless, that is practical only in the simplest cases. Another possibility is the extraction of the imaginary part from the solution of the differential equations, which of course requires the knowledge of the solution itself. More interestingly, one can observe that often the differential equations become substantially simpler when restricted to the imaginary part only, so that their solution can become easier.

In any case, once the imaginary part of some amplitude $A(d;u)$, say $\mathrm{Im}A(d;u)$, is obtained, one has at disposal the dispersive representation for $A(d;u)$, namely an expression of the form

$$A(d;u) = \frac{1}{\pi}\int dt\ \mathrm{Im}A(d;t)\ \frac{1}{t-u}$$

(where the limits of integration have been skipped for ease of typing). Such a representation turns out to be very useful when the amplitude $A(d;u)$ appears within the inhomogeneous terms of some other differential equation, regardless of the actual analytical expression of $A(d;u)$. Indeed, as the whole dependence on u is in the denominator $(t-u)$ one can work out its contribution by considering only that denominator, freezing, so to say, the t-integration and the weight $\mathrm{Im}A(d;t)$ until the dependence on the variable u (the variable of the differential equation) has been properly processed. Let us emphasize, again, that such a processing is, obviously, fully independent of the actual form of $\mathrm{Im}A(d;t)$.

In the following, we will illustrate the above remarks in a couple of elementary applications and then use them in the case of the two-loop QED-kite, i.e. the two-loop electron self-mass in QED, already studied by Sabry [25] long ago. The study of the kite amplitudes requires in turn the knowledge of the two-loop massive sunrise, which appears as inhomogeneous terms in their differential equations. Indeed, the imaginary part [26] and related dispersion relations [27,28] have been already exploited long ago for studying the zeroth order of the sunrise and the kite integral. In this paper our goal is more general, as we will show how to use them consistently within the differential equations approach, which will allow us to investigate the solution at any order in the $(d-4)$ expansion.

The paper is organized as follows. We begin in section 2 studying the imaginary part of the one-loop self mass and its dispersion relation for generic values of the dimensions d. We elaborate on its calculation both from Cutkosky–Veltman rule and from the differential equations. In section 3 we study a particular vertex amplitude through the differential equations method. The one-loop self-mass appears as inhomogeneous term in the equations and we show that their evaluation can be simplified, once the one-loop self-mass is inserted as dispersive relation. In section 4