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The $SU(r)_2$ string functions as q-diagrams

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Abstract

A generalized Roger Ramanujan (GRR) type expression for the characters of A-type parafermions has been a long standing puzzle dating back to conjectures made regarding some of the characters in the 90s. Not long ago we have put forward such GRR type identities describing any of the level two ADE-type generalized parafermions characters at any rank. These characters are the string functions of simply laced Lie algebras at level two, as such, they are also of mathematical interest. In our last joint paper we presented the complete derivation for the D-type generalized parafermions characters identities. Here we generalize our previous discussion and prove the GRR type expressions for the characters of A-type generalized parafermions. To prove the A-type GRR conjecture we study further the q-diagrams, introduced in our last joint paper, and examine the diagrammatic interpretations of known identities among them Slater identities for the characters of the first minimal model, which is the Ising model, and the Bailey lemma.

1. Introduction

Two dimensional theories comprised of a matter content which includes generalized parafermions have been the subject of many papers along the years. A prominent example that has attracted a lot of interest since the emergence of the AGT correspondence [1] is the N-th affine para-Toda theory [2,3]. First it was realized in [4–6] that CFTs with affine and W_k-symmetry are related to the instanton counting for the $SU(N)$ gauge group of rank $r = N - 1$.

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This led the authors of [7] to extend the correspondence between instanton partition functions and conformal blocks of two-dimensional CFT's to the case of $N = 1$ SUSY. In particular, it was found that the instanton partition function of the $SU(2)$ Yang Mills theory evaluated on the $\mathbb{Z}_2$ symmetric instanton moduli space is related to super Liouville conformal blocks in the Whittaker limit. Interestingly, the symmetries of this model are the affine $SU(2)$ at level two and the super-Virasoro symmetries. In search of further generalizations of the 4-d instanton partition function 2-d CFT correspondence web a new idea was proposed in [8]. Following M-theory interpretation of two $M5$-branes on $\mathbb{R}_4/\mathbb{Z}_2$ it was suggested that k $M5$-branes on $\mathbb{R}_4/\mathbb{Z}_N$ realize a $2d$ theory with a free boson, the affine $SU(N)_k$, and the N-th para-W_k symmetry. Naturally, this includes the standard W_k symmetry for $N = 1$ and super-Virasoro symmetry for $k = N = 2$ just mentioned. Finally, the N-th para-W_k algebra is the symmetry of the N-th para-Toda model of type $SU(N)$, which has the action [9]

$$S(\frac{SU(r+1)_k}{U(1)^r}) + \int d^2x[\partial_u \Phi \partial_u \Phi + \sum_{i=0}^{r} \Psi^0_{\alpha_i} \bar{\Psi}^0_{\alpha_i} exp(\frac{b}{\sqrt{k}} \alpha_i \cdot \Phi)] \tag{1.1}$$

where Φ is a vector of r boson fields, α_i are the simple roots of the affine $SU(r+1)_k$, b is related to the background charge and $S(\frac{SU(r+1)_k}{U(1)^r})$ stands for the formal action of the generalized parafermions Ψ^{Λ}_{λ}. As is implied above, generalized $G(r)_k$ type parafermions[1] Ψ^{Λ}_{λ} are more generally defined as describing the excitations associated to the

$$H(G(r)_k) = \frac{G(r)_k}{U(1)^r} \tag{1.2}$$

coset CFT [10] where, for our purposes, $G(r)_k = A(r)_k$, $D(r)_k$, $E(r)_k$ is any of the simply laced affine Lie algebras of rank r and level k.

A daunting problem in the study of such theories, which include generalized parafermions in the matter content, is describing their partition functions and in particular the characters associated with the primary generalized parafermions. Indeed, until recently, the characters corresponding to the generalized parafermion primaries were actually unknown. In a series of articles initiated by one of the authors and A. Belavin these have gradually been uncovered. First, the characters of $SU(N)_2$ generalized parafermions were found in [12] via the ladder coset construction. Interestingly, it was shown that the A type parafermions theories of level two, at any rank, can be realized by a product theory of minimal models with particular combinations of the representing fields taken to insure modular invariance is preserved. This was followed by [13] where this program was generalized to all simply laced affine Lie algebras and the ADE generalized parafermions characters of level two at any rank were also found. More specifically, the authors of [12] considered the coset

$$\mathcal{A}(k, r) = \mathcal{H} \times SU(r)_k \times \frac{SU(k)_r \times SU(k)_n}{SU(k)_{r+n}} \tag{1.3}$$

corresponding to the construction described above for the k $M5$-branes on $\mathbb{R}_4/\mathbb{Z}_N$ instanton partition function. Where $\mathcal{H}$ stands for the Heisenberg algebra and n is given in terms of the Nekrasov parameters $\epsilon_{1,2}$ [14] as follows:

$$\epsilon_1 = n + r, \qquad \epsilon_2 = k - n - r. \tag{1.4}$$

[1] These were separately developed in mathematics as Z algebras [11].