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Thermodynamics of Inozemtsev's elliptic spin chain

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Abstract

We study the thermodynamic behaviour of Inozemtsev's long-range elliptic spin chain using the Bethe ansatz equations describing the spectrum of the model in the infinite-length limit. We classify all solutions of these equations in that limit and argue which of these solutions determine the spectrum in the thermodynamic limit. Interestingly, some of the solutions are not selfconjugate, which puts the model in sharp contrast to one of the model's limiting cases, the Heisenberg XXX spin chain. Invoking the string hypothesis we derive the thermodynamic Bethe ansatz equations (TBA-equations) from which we determine the Helmholtz free energy in thermodynamic equilibrium and derive the associated Y-system. We corroborate our results by comparing numerical solutions of the TBA-equations to a direct computation of the free energy for the finite-length hamiltonian. In addition we confirm numerically the interesting conjecture put forward by Finkel and González-López that the original and supersymmetric versions of Inozemtsev's elliptic spin chain are equivalent in the thermodynamic limit.

1. Introduction

The Bethe ansatz has been one of the most powerful tools in the field of integrability in the past eighty years. Its origin dates back to Bethe's solution of the Heisenberg model for the ferromagnetic interaction of electrons from 1931 [1]. Since then, analysis of numerous models other than spin chains benefited greatly from this ansatz, including the one-dimensional Bose gas [2], two-dimensional lattice models such as the six-vertex model [3] and even $\mathcal{N} = 4$ super

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Yang–Mills theory [4,5]. Moreover, many extensions of Bethe ansatz have been found, including the thermodynamic Bethe ansatz [6,7], nested Bethe ansatz [8] and asymptotic Bethe ansatz [9–11].

Heisenberg's spin-$1/2$ XXX spin chain is still ubiquitous in the research field centred around the Bethe ansatz. In an effort to generalize this spin chain, Inozemtsev proposed an elliptic spin chain characterized by the hamiltonian

$$H = -\frac{J}{8} \sum_{\substack{j,k=1 \\ j \neq k}}^{L} \wp_L(j-k) \left(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1 \right), \tag{1}$$

where L is the number of sites of the spin chain, J is the interaction parameter and $\wp_L$ is the Weierstraß elliptic function with periods $(L, i\pi/\kappa)$ (for $\kappa > 0$) (see Appendix A) and $\boldsymbol{\sigma}$ is the usual vector of Pauli spin-$1/2$ operators [12]. Amazingly, this spin chain not only generalizes the Heisenberg XXX spin chain, which is recovered by taking $\kappa \to \infty$, but actually interpolates smoothly between the (nearest-neighbour) XXX spin chain and the long-range Haldane–Shastry spin chain, obtained in the limit $\kappa \to 0$. The Haldane–Shastry spin chain is solvable by exploiting its Yangian symmetry already present at finite length [13,14]. Therefore, investigating Inozemtsev's elliptic spin chain may shed light on the relation between these two methods for finding exact solutions. In particular, the integrability of both the Heisenberg XXX spin chain and the Haldane–Shastry spin chain suggest that Inozemtsev's elliptic spin chain might also be integrable. Although a definite proof remains absent to date, research into this question has culminated in a proposed set of L conserved quantities [15] and a description of eigenstates at finite and infinite L, which were found using an extended version of Bethe ansatz [16]. Another piece of evidence interestingly comes from the analysis of the level density of the spectrum of the spin chain, which agrees to great accuracy with some existing conjectures about chaotic versus integrable behaviour of quantum systems [17].

In fact, the spectrum of Inozemtsev's elliptic spin chain has been studied before. Dittrich and Inozemtsev probed the spectrum of Inozemtsev's infinite-length spin chain by classifying its two-particle bound states [18]. Later, this spin chain was also used in a completely different context to calculate the first corrections to the dilatation operator in $\mathcal{N} = 4$ super Yang–Mills theory. To this end asymptotic Bethe ansatz for Inozemtsev's spin chain was used to calculate corrections to the spectrum of the Heisenberg XXX spin chain as a truncated power series in κ [5], thereby providing some perturbative results on the spectrum of Inozemtsev's elliptic spin chain in the large volume limit.

Finally, the related supersymmetric su(1|1) version of Inozemtsev's elliptic spin chain was studied in [19] and shown to be integrable. It interpolates smoothly between the supersymmetrizations of the Heisenberg XXX spin chain (the XX spin chain at critical strength of the magnetic field) and of the Haldane–Shastry spin chain [20,21]. The thermodynamic limit of the su(1|1) elliptic spin chain was studied and shown to correctly reproduce the behaviour of the aforementioned models in the appropriate limits. In addition, the Heisenberg XXX and Haldane–Shastry spin chain turn out to be equivalent to their supersymmetrizations in the thermodynamic limit and it has been hypothesized that this equivalence also carries over to the elliptic spin chain.

In this work, we aim to gain additional information about the spectrum in the thermodynamic limit by invoking the string hypothesis [1], i.e. by assuming that the solutions of the Bethe ansatz equations in the infinite-length limit completely describe the thermodynamic behaviour of the