Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B 907 (2016) 258–285
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

S-functions, spectral functions of hyperbolic geometry, and vertex operators with applications to structure for Weyl and orthogonal group invariants

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Received 22 February 2016; accepted 23 March 2016

Available online 30 March 2016

Editor: Hubert Saleur

Dedicated to the memory of our friend and colleague, Petr P. Kulish

Abstract

In this paper we analyze the quantum homological invariants (the Poincaré polynomials of the $\mathfrak{sl}_N$ link homology). In the case when the dimensions of homologies of appropriate topological spaces are precisely known, the procedure of the calculation of the Kovanov–Rozansky type homology, based on the Euler–Poincaré formula can be appreciably simplified. We express the formal character of the irreducible tensor representation of the classical groups in terms of the symmetric and spectral functions of hyperbolic geometry. On the basis of Labastida–Mariño–Ooguri–Vafa conjecture, we derive a representation of the Chern–Simons partition function in the form of an infinite product in terms of the Ruelle spectral functions (the cases of a knot, unknot, and links have been considered). We also derive an infinite-product formula for the orthogonal Chern–Simons partition functions and analyze the singularities and the symmetry properties of the infinite-product structures.

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http://dx.doi.org/10.1016/j.nuclphysb.2016.03.029

1. Introduction

Graded Poincaré polynomials; infinite-dimensional algebras. The aim of this paper is to exploit and emphasize the structure of quantum group invariants. Recall some recent activities related to quantum invariants and the homological invariants of the Hopf link.

Note a certain importance in many diverse areas of mathematics and physics a class of infinite-dimensional algebras, in particular (affine) Kac–Moody algebras, which has been introduced in the late 1960s. Unlike the finite-dimensional case (where simple Lie algebras can be realized in terms of a finite number of bosonic/fermionic *modes*), infinite-dimensional Kac–Moody algebras have various vertex operator realizations (in terms of a finite number of bosonic *free fields*, the modes of which generate a Heisenberg algebra). Generally speaking, all simple (twisted and untwisted) Kac–Moody algebras can be embedded in the infinite-dimensional algebra $\mathfrak{gl}(\infty)$ of infinite matrices with a finite number of non-zero entries (see Sect. 3), which has a simple realization in terms of generators of a Clifford algebra.

Virasoro algebra as an another type of infinite-dimensional algebra arises in different areas of physics (see Sect. 3 for details); this is the algebra of conformal transformations in two-dimensions. The operator algebra structure of two-dimensional conformally-invariant quantum field theories is determined by the representation theory of the Virasoro algebra. The Ramond and Neveu–Schwarz superalgebras (or the $N = 1$ superconformal algebras) are supersymmetric extensions of the Virasoro algebra.

We also mention the quantum affine algebras, which are q-deformations of Kac–Moody algebras. By analogy with the undeformed case, vertex operator realizations, initially for level one representations and then for arbitrary high level representations, were constructed.

Symmetric functions (or S-functions) with it connection to replicated plethysms, has been involved in the applications to certain infinite-dimensional Lie algebras (in particular the quantum affine algebras $U_q(\mathfrak{gl}_N)$) and generating functions of quantum $\mathfrak{gl}_N$ invariants. The S-functions were first studied by Jacobi and have been generalized in various applications in physics and mathematics. There are numerous generalizations of S-functions, and among them Jack symmetric functions. Jack symmetric functions are just a special limit of a generalized Hall–Littlewood function considered by Macdonald [1]. Macdonald's polynomials can be formulated as the trace of an interwiner (algebra homomorphism) of modules over the quantum group [2].

Quantum group invariants; finite-dimensional algebras. Another type of algebra, which has had a wide variety of applications in physics, is the so-called quantum group; this may be regarded as a deformation, depending on a parameter q, of the universal enveloping algebra of a semi-simple Lie algebra. Thus they are not finite-dimensional algebras, but are finitely generated. These algebras were first constructed by Kulish and Reshetikhin [3] and as a Hopf algebra by Sklyanin [4]. Their representation theory for q not a root of unity was found to be similar to the corresponding semi-simple Lie algebra, complications arise when q is a root of unity due to the fact that the centre of the algebra becomes larger. Quantum groups are an example of quasi-triangular Hopf algebras and as such, for each quantum group there exists a universal R-matrix which intertwines with the action of the coproduct.

By exploiting calculations for oriented and unoriented links the two-variable HOMFLY and one-variable Kauffman polynomials have been analyzed in [5,6] and [7] respectively. Note that HOMFLY polynomial can be generalized to Alexander and Jones polynomial; the two-variable Kauffman polynomial has been also introduced in [8] by generalization of the Jones polynomial. A polynomial invariant of oriented knots has been discovered in [9,10]. A quantum field theory