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Nuclear Physics B 907 (2016) 479–494

NUCLEAR PHYSICS B

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Algebraic geometry methods associated to the one-dimensional Hubbard model

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Received 3 February 2016; accepted 11 April 2016

Available online 16 April 2016

Editor: Hubert Saleur

Abstract

In this paper we study the covering vertex model of the one-dimensional Hubbard Hamiltonian constructed by Shastry in the realm of algebraic geometry. We show that the Lax operator sits in a genus one curve which is not isomorphic but only isogenous to the curve suitable for the AdS/CFT context. We provide an uniformization of the Lax operator in terms of ratios of theta functions allowing us to establish relativistic like properties such as crossing and unitarity. We show that the respective R-matrix weights lie on an Abelian surface being birational to the product of two elliptic curves with distinct J-invariants. One of the curves is isomorphic to that of the Lax operator but the other is solely fourfold isogenous. These results clarify the reason the R-matrix can not be written using only difference of spectral parameters of the Lax operator.

1. Introduction

The Hubbard model originates from the tight-binding formulation for solids where the electrons can hop between lattice sites but also interact through the Coulomb repulsion. In its simplest form, electron hopping takes place between nearest neighbor sites with the same kinetic energy while the Coulomb interaction occurs only for electrons at the same site with a constant

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http://dx.doi.org/10.1016/j.nuclphysb.2016.04.019

strength U. The Hubbard Hamiltonian on a ring of size N with interaction symmetric under electron-hole transformation is given by,

$$\mathrm{H} = -\sum_{j=1}^{\mathrm{N}} \sum_{\sigma=\uparrow,\downarrow} (c_{j\sigma}^{\dagger} c_{j+1\sigma} + c_{j+1\sigma}^{\dagger} c_{j\sigma}) + \mathrm{U} \sum_{j=1}^{\mathrm{N}} (c_{j\uparrow}^{\dagger} c_{j\uparrow} - \frac{1}{2})(c_{j\downarrow}^{\dagger} c_{j\downarrow} - \frac{1}{2}), \tag{1}$$

where $c_{j\sigma}^{\dagger}$ and $c_{j\sigma}$ stand for creation and annihilation operators for an electron at site j with spin σ.

In a groundbreaking work Lieb and Wu showed that Hamiltonian (1) is exactly diagonalized by means of an extension of the coordinate Bethe ansatz method besides the model absence of Mott transition [1]. Over the years this solution has been used to compute many other physical properties and for a recent extensive review we refer to the monograph [2]. Exact integrability from the viewpoint of the quantum inverse scattering approach was only established many years later by Shastry in three influential papers [3–5]. An important result was the discovery of a classical two-dimensional vertex model on the square $\mathrm{N} \times \mathrm{N}$ lattice whose row-to-row transfer matrix commutes with the spin version of the Hubbard Hamiltonian. This spin model was obtained by applying a generalized version of the Jordan–Wigner transformation on the bulk term of Eq. (1) which can be rewritten as [3],

$$\mathrm{H} = \sum_{j=1}^{\mathrm{N}} \sigma_j^{+} \sigma_{j+1}^{-} + \sigma_j^{-} \sigma_{j+1}^{+} + \tau_j^{+} \tau_{j+1}^{-} + \tau_j^{-} \tau_{j+1}^{+} + \frac{\mathrm{U}}{4} \sigma_j^{z} \tau_j^{z}, \tag{2}$$

where $\sigma_j^{\pm}, \sigma_j^{z}$ and $\tau_j^{\pm}, \tau_i^{z}$ are two commuting sets of Pauli matrices acting on the site j. Recall that strict periodic boundary conditions for electron Hamiltonian (1) lead to sector dependent twisted boundary conditions for the spin operator (2) and the precise form of this relationship can for instance be found in [6]. However, this difference on boundaries can be easily captured by introducing fermionic statistics into the integrable structures without affecting the main features of Shastry's construction [7].

The appealing form of the spin Hamiltonian (2) led Shastry to propose that the underlying classical vertex model should be given by coupling appropriately two six-vertex models obeying the so-called free-fermion condition. Let us denote by $\mathrm{L}_{0j}(\omega)$ the Lax operator encoding the Boltzmann weights structure of such coupled six-vertex models. As usual the indices 0 and j refer to operators acting on the auxiliary and quantum spaces associated respectively with the degrees of freedom sited on the horizontal and vertical edges of the square lattice. In terms of Pauli matrices such Lax operator can be expressed by,

$$\mathrm{L}_{0j}(\omega) = \exp\left[\frac{h}{2}(\sigma_0^z \tau_0^z + \mathrm{I}_0)\right] \mathrm{I}_j \left[\mathcal{L}_{0j}^{(\sigma)}(a,b,c) \mathcal{L}_{0j}^{(\tau)}(a,b,c)\right] \exp\left[\frac{h}{2}(\sigma_0^z \tau_0^z + \mathrm{I}_0)\right] \mathrm{I}_j, \tag{3}$$

where I is the four-dimensional identity matrix and the symbol ω denotes the set of parameters a, b, c and h.

The Lax operators $\mathcal{L}_{0j}^{(\sigma)}(a,b,c)$ and $\mathcal{L}_{0j}^{(\tau)}(a,b,c)$ represent the weights of two copies of six-vertex models whose expressions are,

$$\mathcal{L}_{0j}^{\sigma}(a,b,c) = \frac{(a+b)}{2} \mathrm{I}_0 \mathrm{I}_j + \frac{(a-b)}{2} \sigma_0^z \sigma_j^z + c(\sigma_0^{+} \sigma_j^{-} + \sigma_0^{-} \sigma_j^{+}), \tag{4}$$

and

$$\mathcal{L}_{0j}^{\tau}(a,b,c) = \frac{(a+b)}{2} \mathrm{I}_0 \mathrm{I}_j + \frac{(a-b)}{2} \tau_0^z \tau_j^z + c(\tau_0^{+} \tau_j^{-} + \tau_0^{-} \tau_j^{+}), \tag{5}$$