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Generalized spin Sutherland systems revisited

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Abstract

We present generalizations of the spin Sutherland systems obtained earlier by Blom and Langmann and by Polychronakos in two different ways: from $SU(n)$ Yang–Mills theory on the cylinder and by constraining geodesic motion on the N-fold direct product of $SU(n)$ with itself, for any $N > 1$. Our systems are in correspondence with the Dynkin diagram automorphisms of arbitrary connected and simply connected compact simple Lie groups. We give a finite-dimensional as well as an infinite-dimensional derivation and shed light on the mechanism whereby they lead to the same classical integrable systems. The infinite-dimensional approach, based on twisted current algebras (alias Yang–Mills with twisted boundary conditions), was inspired by the derivation of the spinless Sutherland model due to Gorsky and Nekrasov. The finite-dimensional method relies on Hamiltonian reduction under twisted conjugations of N-fold direct product groups, linking the quantum mechanics of the reduced systems to representation theory similarly as was explored previously in the $N = 1$ case.

1. Introduction

The Calogero–Moser–Sutherland type many-body systems [1–3] and their generalizations continuously attract attention since they are ubiquitous in physical applications and are related

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to intriguing mathematical structures. See, for instance, the reviews [4–7]. A powerful approach to these integrable systems consists in viewing them as Hamiltonian reductions of higher dimensional free systems. This method first came to light in the papers [8,9]. In particular, Kazhdan, Kostant and Sternberg [9] obtained the classical Sutherland Hamiltonian

$$\mathcal{H}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{i\neq j} \frac{\nu^2}{\sin^2(q_i - q_j)} \tag{1.1}$$

by reducing the kinetic energy of the geodesic motion on the group $SU(n)$ using the symmetry defined by the conjugation action of $SU(n)$ on itself. Their reduction required fixing the Noether charges of the symmetry in a very special manner. Later it turned out that more general choices lead to extensions of the Sutherland system by 'spin' degrees of freedom that belong to reductions of coadjoint orbits of $SU(n)$ by the action of the maximal toral subgroup. Spin Sutherland systems of this kind appear for every simple Lie group [5,10–12]. Moreover, not only the choice of the constraints, alias the value of the momentum map, but also the underlying symmetry can be generalized in several ways. For example, we obtained spin Sutherland systems by reducing the free motion on simple compact Lie groups utilizing twisted conjugations [13]. These reductions were analyzed [14] at the quantum mechanical level, too, which yields a bridge from harmonic analysis to integrable systems.

The derivation of the Calogero and Sutherland systems given in [9] represents a paradigm for the reduction approach to integrable systems. This framework has since been expanded by the introduction of several new parent systems as master integrable systems. We mention only the influential work of Gorsky and Nekrasov [15,16], where the Sutherland Hamiltonian was re-derived from a free Hamiltonian on an infinite-dimensional phase space built on $su(n)$-valued currents on the circle. As discussed also in [5,17], this connects the Sutherland system to $SU(n)$ Yang–Mills theory on the cylinder.

Our present work was motivated mainly by questions stemming from the papers of Blom and Langmann [18,19] and Polychronakos [20], where a family of generalized spin Sutherland systems was derived by means of two different methods. In these systems the particle positions are coupled to spin variables living on N arbitrary coadjoint orbits of $SU(n)$, and the Hamiltonian also involves N arbitrary scalar parameters, for any integers $n > 1$ and $N > 1$. Blom and Langmann obtained their systems from $SU(n)$ Yang–Mills theory on the cylinder, placing non-dynamical 'color charges' at N arbitrary locations on the circle. Thus their derivation fits in the Gorsky–Nekrasov framework. On the other hand, Polychronakos proceeded by constraining the geodesic motion on the N-fold direct product of $SU(n)$ with itself, using N arbitrary scale factors in the definition of the bi-invariant Riemannian metric. Although it was not stressed in [20], this amounts to Hamiltonian symmetry reduction with respect to twisted conjugations on the direct product group, defined with the aid of the cyclic permutation of the N-factors. Based on direct comparison of the Hamiltonians, both Blom–Langmann [19] and Polychronakos [20] pointed out that their respective systems coincide, but they did not provide any conceptual explanation of this remarkable fact. It is natural to ask for a better understanding of the mechanism behind this coincidence.

In this paper we describe two derivations of group theoretic generalizations of the systems of [18,20] and shed light on the mechanism whereby these two derivations lead to the same outcome. The first approach relies on symplectic reduction of geodesic motion on the N-fold direct product $G \times \cdots \times G$ for any compact simple Lie group G. The underlying symmetry is built from twisted conjugations involving the cyclic permutation of the N factors and, for simply laced