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Static knot energy, Hopf charge, and universal growth law

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Abstract

We present a family of static knotted soliton energy functionals governing the configuration maps from the Euclidean space $\mathbb{R}^{4n-1}$ into the unit sphere S^{2n} so that the knot charges are naturally represented by the Hopf invariants in the homotopy group $\pi_{4n-1}(S^{2n})$ and the special case $n = 1$ recovers the classical Faddeev knot energy. We establish the general result that the minimum energy or the knot mass E_N of knotted solitons of the Hopf charge N satisfies the universal fractional-exponent growth law $E_N \sim |N|^{(4n-1)/4n}$, in which the fractional exponent depends only on the dimensions of the domain and range spaces of the configuration maps but does not depend on the detailed structure of the knot energy.

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1. Introduction

It was Lord Kelvin who initiated the study of knots and explored the idea that atoms could be thought of made of knotted vortex tubes of ether so that the stability of matter could be explained by the topological stability of knots, the variety of chemical elements could be viewed as a consequence of the variety of knots, and the spectral lines of atoms could be considered as

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a reflection of the oscillatory patterns of knots [1]. Although such an idea is now known to be incorrect, it has inspired various fundamental areas in modern science. For example, knots may be used to explain the concept of spin [2]; elementary particles may be regarded as quantized flux loops represented by knots or links and antiparticles by their mirror images [3]; knotted cosmic strings may be produced in the early stages of the universe which would be responsible for the initial matter accretion for galaxy formation [4,5]; knotted structures may appear in ferromagnetic spin-triplet superconductors [6]; the topological classification of knots may give clues to various aspects of DNA [7]; the entanglement structures of polymers may also be understood based on a knowledge on knots [8]. Over the last one hundred years, numerous contributions to the classification of knots have been made. Notably, Tait [9] enumerated knots in terms of the crossing number of a plane projection; Alexander [10] discovered a knot invariant, known as the Alexander polynomial, arising in 3-dimensional homology; Jones [11] found a new knot invariant, known as the Jones polynomial, which enabled several conjectures of Tait to be proved [12]; based on a heuristic quantum-field theory argument, Witten [13] derived from the Chern–Simons action a family of knot invariants including the Jones invariant; finally came the Vassiliev invariants [14] which cover the Alexander polynomial and the Jones polynomial and lay a general framework for the study of the combinatorial aspects of knots.

Recently, Faddeev and Niemi [15] used computer simulation, relaxation and toroidal coordinates to show that a ring-shaped (unknotted) Hopf charge one soliton exists as the energy minimizer of a relativistic quantum field theory model proposed many years ago by Faddeev [16], which may be viewed [17] as a refined Skyrme model [18] modeling mesons and baryons. A more extensive computer investigation was later conducted by Battye and Sutcliffe [19] and a variety of knotted solitons were obtained. In normalized form, the energy governing the static knotted configurations in the Faddeev model [15,16,19,20] over the Euclidean 3-space $\mathbb{R}^3$ is

$$E(\mathbf{n}) = \int_{\mathbb{R}^3} \left\{ \sum_{1\leqslant j\leqslant 3} |\partial_j \mathbf{n}|^2 + \frac{1}{2} \sum_{1\leqslant j<k\leqslant 3} \left|F_{jk}(\mathbf{n})\right|^2 \right\} \mathrm{d}x, \tag{1.1}$$

where the field $\mathbf{n} = (n_1, n_2, n_3)$ assumes its values in the unit 2-sphere and $F_{jk}(\mathbf{n}) = \mathbf{n}\cdot(\partial_j \mathbf{n} \wedge \partial_k \mathbf{n})$ $(j,k = 1,2,3)$. The finite-energy condition implies that $\mathbf{n}$ approaches a constant vector at spatial infinity of $\mathbb{R}^3$. Hence we may compactify $\mathbb{R}^3$ into S^3 and view the fields as maps from S^3 to S^2. As a consequence, we see that each finite-energy field configuration $\mathbf{n}$ is associated with an integer, $Q(\mathbf{n})$, in $\pi_3(S^2) = \mathbb{Z}$. In fact, such an integer $Q(\mathbf{n})$ is known as the Hopf invariant which has the following integral characterization due to Whitehead [21]: since the vector field $\mathbf{F} = (\frac{1}{2}\epsilon^{jk\ell} F_{k\ell}(\mathbf{n}))$ is divergence free, there is a vector potential $\mathbf{A}$ so that $\mathbf{F} = \nabla \wedge \mathbf{A}$. In terms of $\mathbf{A}$ and $\mathbf{F}$, the Hopf charge $Q(\mathbf{n})$ of the map $\mathbf{n}$ may then be evaluated by the integral

$$Q(\mathbf{n}) = \frac{1}{16\pi^2} \int_{\mathbb{R}^3} \mathbf{A}\cdot\mathbf{F}\,\mathrm{d}x, \tag{1.2}$$

which is also a Chern–Simons invariant [22]. Mathematically, the knotted solitons of the Faddeev model are the solutions to the minimization problem [23]

$$E_N = \inf\left\{E(\mathbf{n}) \mid E(\mathbf{n}) < \infty,\ Q(\mathbf{n}) = N\right\}, \qquad N \in \mathbb{Z}. \tag{1.3}$$

It has been recognized [24] that the crucial characteristic that guarantees the existence of energy minimizing Faddeev knots is the sublinear energy growth law

$$C_1|N|^{3/4} \leqslant E_N \leqslant C_2|N|^{3/4}, \tag{1.4}$$