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Towards all-order Laurent expansion of generalised hypergeometric functions about rational values of parameters

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Abstract

We prove the following theorems: (1) the Laurent expansions in ε of the Gauss hypergeometric functions ${}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{p}{q} + c\varepsilon; z)$, ${}_2F_1(I_1 + \frac{p}{q} + a\varepsilon, I_2 + \frac{p}{q} + b\varepsilon; I_3 + \frac{p}{q} + c\varepsilon; z)$ and ${}_2F_1(I_1 + \frac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + \frac{p}{q} + c\varepsilon; z)$, where I_1, I_2, I_3, p, q are arbitrary integers, a, b, c are arbitrary numbers and ε is an infinitesimal parameter, are expressible in terms of multiple polylogarithms of q-roots of unity with coefficients that are ratios of polynomials; (2) the Laurent expansion of the Gauss hypergeometric function ${}_2F_1(I_1 + \frac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z)$ is expressible in terms of multiple polylogarithms of q-roots of unity times powers of logarithm with coefficients that are ratios of polynomials; (3) the multiple inverse rational sums $\sum_{j=1}^{\infty} \frac{\Gamma(j)}{\Gamma(1+j-\frac{p}{q})} \frac{z^j}{j^c} S_{a_1}(j-1) \times \cdots \times S_{a_p}(j-1)$ and the multiple rational sums $\sum_{j=1}^{\infty} \frac{\Gamma(j+\frac{p}{q})}{\Gamma(1+j)} \frac{z^j}{j^c} S_{a_1}(j-1) \times \cdots \times S_{a_p}(j-1)$, where $S_a(j) = \sum_{k=1}^{j} \frac{1}{k^a}$ is a harmonic series and c is an arbitrary integer, are expressible in terms of multiple polylogarithms; (4) the generalised hypergeometric functions ${}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon, \frac{p}{q} + B_{p-1}; z)$ and ${}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon, \frac{p}{q} + A_p; \vec{B} + \vec{b}\varepsilon; z)$ are expressible in terms of multiple polylogarithms with coefficients that are ratios of polynomials with complex coefficients.

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Keywords: Gauss hypergeometric functions; Generalised hypergeometric functions; Laurent expansion about rational values of parameters; Multiple polylogarithms; Multiloop calculations; Two-loop sunset

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1. Introduction: Feynman diagrams and special functions

High-precision theoretical predictions for physics at the LHC and the ILC demand the inclusion of higher-order radiative corrections. The results of perturbative calculation are expressible in terms of Feynman integrals [1]. However, in order to obtain physical results, it is necessary to construct the Laurent expansions of Feynman diagrams about the integer value of the space–time dimension [2] (typically $d = 4 - 2\varepsilon$). For the parametrisation of the coefficients of such ε expansions, a lot of new functions have been introduced by physicists during the last few years [3–6]. Some of these new functions are also generated in a different branch of mathematics [7–10]. At present, it is unclear if there is some limitation on the types of functions generated by Feynman diagrams or if the "zoo" of new functions is an artifact of the techniques used. In particular, the statement that the results of such calculations can be written in terms of a restricted set of special functions will allow one to use a restricted set of programs for the numerical evaluation of physical results. Another application is related to the evaluation of so-called single-scale diagrams, where an explicit prediction of possible transcendental constants can be done [11].

The strategy of such a kind of analysis is well known in the theory of special functions and the analytical theory of differential equations [12]. As is well known, any Feynman diagram satisfies a system of linear differential or difference equations with polynomial coefficients [13–17]. In modern mathematical language, such a system can be associated with the Gelfand–Karpanov–Zelevinskii functions or D-modules [18]. So, any question regarding the zoo of special functions generated by the ε expansion of Feynman diagrams may be reduced to the problem of constructing Laurent expansions of D-modules (hypergeometric functions [19]) about certain values of their parameters. Unfortunately, a unique hypergeometric representation of Feynman diagrams besides the so-called α representation [1] does not exist. Using the latter representation, it has been shown recently that, for single-scale diagrams, i.e., diagrams where all kinematic variables are proportional to each other so that one of them can be factored out, all coefficients of the ε expansion can be understood, up to some normalisation factor, as periods in the Kontsevich–Zagier formulation [20]. Another useful representation, which is closely related to the corresponding property of generalised hypergeometric functions, is the Mellin–Barnes representation of Feynman diagrams [21]. Using the Mellin–Barnes representation, it is possible to write the result in each order of ε in terms of multiple sums, which sometimes can be expressed in terms of special functions [22]. Since the power of a propagator is integer in covariant gauge and any (irreducible) numerator is expressible in terms of an integral of the same topology with a shifted power, which is again integer [14,15], it is enough to only consider hypergeometric functions of several variables with integer values of parameters. (In general, the number of variables is equal to the number of kinematic invariants minus one.) Fortunately, when some of the kinematic invariants are proportional (or equal) to each other, the number of variables in the proper hypergeometric series can be reduced. But the price of this reduction is the appearance of rational values of parameters. All known exactly solvable cases [23–27] have confirmed this observation. Typically, only integer and half-integer values of parameters are generated, and only recently inverse cubic values have been discovered [27].

Recently, there has been essential progress in understanding what type of functions are generated by the ε expansion of hypergeometric functions. Besides the pioneering constructions of the ε expansions of hypergeometric functions [28] using harmonic series [29] or so-called multiple (inverse) binomial sums [30–32], there are now a few independent techniques for the construction of analytical coefficients in the ε expansions of hypergeometric functions about integer and half-integer values of parameters and the summing of multiple series [32–41]. However, the ex-