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Nuclear Physics B 843 [PM] (2011) 362–381
NUCLEAR PHYSICS B
www.elsevier.com/locate/nuclphysb

Jack polynomial fractional quantum Hall states and their generalizations

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Received 29 July 2010; accepted 27 September 2010

Available online 8 October 2010

Abstract

In the study of fractional quantum Hall states, a certain clustering condition involving up to four integers has been identified. We give a simple proof that particular Jack polynomials with $\alpha = -(r-1)/(k+1)$, $(r-1)$ and $(k+1)$ relatively prime, and with partition given in terms of its frequencies by $[n_0 0^{(r-1)s} k 0^{r-1} k 0^{r-1} k \cdots 0^{r-1} m]$ satisfy this clustering condition. Our proof makes essential use of the fact that these Jack polynomials are translationally invariant. We also consider nonsymmetric Jack polynomials, symmetric and nonsymmetric generalized Hermite and Laguerre polynomials, and Macdonald polynomials from the viewpoint of the clustering.

Keywords: Fractional quantum Hall states; Jack polynomials

1. Introduction

The symmetric Jack polynomials $P_\kappa(z;\alpha)$, $z := (z_1, \ldots, z_N)$ a coordinate in $\mathbb{C}^N$, α a scalar and κ a partition, are an orthogonal, homogeneous basis for symmetric function generalizing the Schur ($\alpha = 1$) and zonal ($\alpha = 2$) polynomials. They appear in physics in random matrix theory [17, Ch. 12 & 13], [11] and in the study of quantum many body wave functions [17, Ch. 11], [5,6]. Here we will be interested in the latter interpretation.

There are two classes of quantum many body systems for which Jack polynomials are relevant, one involving the $1/r^2$ pair potential in one dimension and the other corresponding to certain

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0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2010.09.018

fractional quantum Hall states. Regarding the former [17, Ch. 11], with the domain a unit circle, the corresponding Schrödinger operator reads

$$H^{(C)} := -\sum_{j=1}^{N} \frac{\partial^2}{\partial \theta_j^2} + \frac{\beta}{4}\left(\frac{\beta}{2} - 1\right) \sum_{1 \leqslant j < k \leqslant N} \frac{1}{\sin^2(\theta_k - \theta_j)/2}, \tag{1}$$

where β parametrizes the coupling. With $z_j = e^{i\theta_j}$ the ground state wave function for (1) is proportional to

$$\psi_0^{(C)}(z) := \left|\Delta(z)\right|^{\beta/2}, \quad \Delta(z) := \prod_{1 \leqslant j < k \leqslant N} (z_j - z_k) \tag{2}$$

and with $\alpha := 2/\beta$, a complete set of eigenfunctions is given in terms of Jack polynomials by [17, Eq. (13.199)]

$$\psi_0^{(C)}(z) z^{-l} P_\kappa(z;\alpha) \quad (l = 0, 1, \ldots) \tag{3}$$

where

$$z^\kappa := z_1^{\kappa_1} z_2^{\kappa_2} \ldots z_N^{\kappa_N} \tag{4}$$

and for $l > 0$ it is required that $\kappa_N = 0$.

Next we will revise how certain fractional quantum Hall states relate to Jack polynomials [5,6]. An infinite family of bosonic fractional quantum Hall states, indexed by a positive integer k, are due to Read and Rezayi [31]. For a system of kN particles, these are defined up to normalization as

$$\psi_{RR}^{(k)} = \operatorname{Sym} \prod_{s=1}^{k} \prod_{1 \leqslant i_s < j_s \leqslant N} (z_{i_s} - z_{j_s})^2 \tag{5}$$

where Sym denotes symmetrization (see (18) below). Note that the kN particles are thus partitioned into k groups of N. Setting $k = 1$ we read off that

$$\psi_{RR}^{(1)} = \prod_{1 \leqslant j < k \leqslant N} (z_j - z_k)^2$$

which is the filling factor $\nu = 1/2$ bosonic Laughlin state. For $k = 2$ it turns out that [31]

$$\psi_{RR}^{(2)} = \operatorname{Pf}\left[\frac{1}{z_k - z_l}\right]_{k,l=1,\ldots,2N} \prod_{1 \leqslant i < j \leqslant 2N} (z_i - z_j), \tag{6}$$

where the diagonal entry is to be replaced by zero if $k = l$, which is the filling factor $\nu = 1$ Moore–Read state [27]. As noted in [31], $\psi_{RR}^{(k)}$ is characterized by the requirements that it be symmetric, and exhibit the factorization property

$$\psi_{RR}^{(k)}(z_1, \ldots, z_{(N-1)k}, \underbrace{z, \ldots, z}_{k\text{ times}}) = \prod_{l=1}^{(N-1)k} (z_l - z)^2 \psi_{RR}^{(k)}(z_1, \ldots, z_{(N-1)k}). \tag{7}$$

It is at this stage the Jack polynomials show themselves. Thus it is a remarkable finding of recent times [5] that (7) is satisfied by

$$\psi_{RR}^{(k)}(z) = P_{(2\delta)^k}(z; -k-1), \tag{8}$$