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Analytical approximation schemes for solving exact renormalization group equations. II Conformal mappings

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Abstract

We present a new efficient analytical approximation scheme to two-point boundary value problems of ordinary differential equations (ODEs) adapted to the study of the derivative expansion of the exact renormalization group equations. It is based on a compactification of the complex plane of the independent variable using a mapping of an angular sector onto a unit disc. We explicitly treat, for the scalar field, the local potential approximations of the Wegner–Houghton equation in the dimension $d = 3$ and of the Wilson–Polchinski equation for some values of $d \in]2, 3]$. We then consider, for $d = 3$, the coupled ODEs obtained by Morris at the second order of the derivative expansion. In both cases the fixed points and the eigenvalues attached to them are estimated. Comparisons of the results obtained are made with the shooting method and with the other analytical methods available. The best accuracy is reached with our new method which presents also the advantage of being very fast. Thus, it is well adapted to the study of more complicated systems of equations.

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In a previous article [1] we presented two analytical approaches for studying the derivative expansion of the exact renormalization group equation (ERGE, for reviews and recent pedagogical introductions see [2,3]). The two methods, based on the commonly used field expansion, were shown to be more efficient than the current approaches [4–6] which implicitly assumed that the simple field expansion converges in $[0, \infty[$ [1] whereas it does not [6]. In the first method, introduced in [7], the infinite-boundary condition is explicitly accounted for via an auxiliary differential equation (ADE) whereas, in the second, the solutions looked for are approximated by generalized hypergeometric functions (HFA). Another method very similar to HFA has almost simultaneously been proposed in [8], it looks for the solutions under the form of Padé approximants. Both three methods work well (though the ADE method has a wider range of application) but they are rather heavy to implement (see Table 1). In the present work we show that a simple conformal mapping onto the unit disc of a suitably chosen angular sector of the complex plane of the independent variable, compactifies the originally infinite integration domain so as to make the series of the field expansion in the new variable convergent on the whole disc of unit radius.

The paper is organized as follows. In Section 1, the principle of the mapping method is introduced with the example of the Wegner–Houghton RG flow equation [9] in the local potential approximation (LPA). For the fixed point solution of this equation in three dimensions, one approximately knows the location of the closest singularity in the complex plane of the independent variable [10]. We show that the best convergence properties provided by the method correspond to the largest angular sector compatible with the analyticity of the solution in the original variable. The calculations of the eigenvalues with the mapping method is shown to be easy and we provide the best estimates ever obtained up to now of the fixed point solution and the eigenvalues. In Section 2, we consider the Wilson–Polchinski RG flow equation [11,12] in the LPA. This equation allows us to illustrate the efficiency of the method for different values of the dimension $2 < d \leqslant 3$. Again, we provide the best results ever obtained up to now in three dimensions. We determine the locations of the critical and multicritical fixed points for $d = 3$, $8/3$ and $5/2$ together with the associated eigenvalues for $d = 3$ and $8/3$ with an excellent accuracy. We pursue the determination of the critical fixed point for values of d very close to 2. (At $d = 2$, the type of solutions we track disappears.) In Section 3, we look at the second order of the derivative expansion $O(\partial^2)$ by considering explicitly the Morris RG equations [13] in three dimensions. These equations are much more difficult to treat than the previous ones, even in the LPA. Nevertheless, we are able to determine both the fixed point and the eigenvalues with an accuracy approaching that obtained with the shooting method [13,14]. An estimate of the subcritical "odd-exponent" is obtained for the first time $O(\partial^2)$. It, however, does not compare favourably with existing estimates [15,16]. We discuss the probable reasons of this disagreement. Finally we summarize and conclude.

1. The Wegner–Houghton flow equation in the LPA

As detailed in [1], to which article the reader is invited to refer for some basical definitions if necessary, the study of the existence of fixed points and of their stability in the derivative expansion of an ERGE, amounts to look for regular solutions in $\phi \in [0, \infty[$ of coupled nonlinear ordinary differential equations (ODE). Here ϕ is the (constant) scalar field. Hence the two boundaries associated to the ODEs under study are (see [1]):

(1) $\phi = 0$ where the symmetry of interest is imposed to the solution,
(2) $\phi = \infty$ where a specific behaviour in approaching this point is imposed to the solution.