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Nuclear Physics B 775 [FS] (2007) 225–282
NUCLEAR PHYSICS B

Analytic theory of the eight-vertex model

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Received 27 September 2006; accepted 20 December 2006

Available online 13 January 2007

Abstract

We observe that the exactly solved *eight-vertex solid-on-solid model* contains an hitherto unnoticed arbitrary field parameter, similar to the horizontal field in the six-vertex model. The parameter is required to describe a continuous spectrum of the *unrestricted* solid-on-solid model, which has an infinite-dimensional space of states even for a finite lattice. The introduction of the continuous field parameter allows us to completely review the theory of functional relations in the eight-vertex/SOS-model from a uniform analytic point of view. We also present a number of analytic and numerical techniques for the analysis of the Bethe ansatz equations. It turns out that different solutions of these equations can be obtained from each other by analytic continuation. In particular, for small lattices we explicitly demonstrate that the largest and smallest eigenvalues of the transfer matrix of the eight-vertex model are just different branches of the same multivalued function of the field parameter.

PACS: 02.30.Ik; 05.30.-d; 11.25.Hf

Keywords: Eight-vertex model; Bethe ansatz; Q-operator; Functional relations

1. Introduction

The powerful analytic and algebraic techniques discovered by Baxter in his pioneering papers [1–4] on the exact solution of the eight-vertex lattice model laid the foundation for many important applications in the theory of integrable systems of statistical mechanics and quantum field theory.

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0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2006.12.021

This paper concerns one of these techniques—the method of functional relations. Over the last three decades, since Baxter's original works [1–4], this method has been substantially developed and applied to a large number of various solvable models. However, the status of this method in the eight-vertex model itself with an account of all subsequent developments has not been recently reviewed. This paper is intended to (partially) fill this gap. Here we will adopt an analytic approach exploiting the existence of an (hitherto unnoticed) continuous field parameter in the solvable eight-vertex solid-on-solid model of Ref. [3].

For the purpose of the following discussion it will be useful to first summarize the key results of [1–4]. Here we will use essentially the same notations as those in [1]. Consider the homogeneous eight-vertex (8V) model on a square lattice of N columns, with periodic boundary conditions. The model contains three arbitrary parameters u, η and $\mathsf{q}=e^{i\pi\tau}$, $\operatorname{Im}\tau>0$, which enter the parametrization of the Boltzmann weights (the parameter q enters as the nome for the elliptic theta-functions). The parameters η and q are considered as constants and the spectral parameter u as a complex variable. We assume that the parameter η is real and positive, $0<\eta<\pi/2$, which corresponds to the disordered regime [5] of the model.

The row-to-row transfer matrix of the model, $\mathbf{T}(u)$, possesses remarkable analytic properties. Any of its eigenvalues, $\mathsf{T}(u)$, is both (i) an entire function of the variable u, and (ii) satisfies Baxter's famous functional equation,

$$\mathsf{T}(u)\mathsf{Q}(u)=\mathsf{f}(u-\eta)\mathsf{Q}(u+2\eta)+\mathsf{f}(u+\eta)\mathsf{Q}(u-2\eta), \tag{1.1}$$

where[1]

$$\mathsf{f}(u)=\big(\vartheta_4(u|\mathsf{q})\big)^N, \tag{1.2}$$

and $\mathsf{Q}(u)$ is an entire quasi-periodic function of u, such that

$$\mathsf{Q}(u+\pi)=\pm(-1)^{N/2}\mathsf{Q}(u),\qquad \mathsf{Q}(u+2\pi\tau)=\mathsf{q}^{-2N}e^{-2iuN}\mathsf{Q}(u). \tag{1.3}$$

These analytic properties completely determine all eigenvalues of the transfer matrix $\mathbf{T}(u)$. Indeed, Eq. (1.1) implies that the zeroes $u_1,u_2,\ldots,u_n$, of $\mathsf{Q}(u)$ satisfy the Bethe ansatz equations,

$$\frac{\mathsf{f}(u_k+\eta)}{\mathsf{f}(u_k-\eta)}=-\frac{\mathsf{Q}(u_k+2\eta)}{\mathsf{Q}(u_k-2\eta)},\qquad \mathsf{Q}(u_k)=0,\ k=1,\ldots,n. \tag{1.4}$$

These equations, together with the periodicity relations (1.3), define the entire function $\mathsf{Q}(u)$ (there will be many solutions corresponding to different eigenvectors). Once $\mathsf{Q}(u)$ is known the eigenvalue $\mathsf{T}(u)$ is evaluated from (1.1).

The entire functions $\mathsf{Q}(u)$ appearing in (1.1) are, in fact, eigenvalues of another matrix, $\mathbf{Q}(u)$, called the $\mathbf{Q}$-matrix. Originally it was constructed [1] in terms of some special transfer matrices. A different, but related, construction of the $\mathbf{Q}$-matrix was given in [2] and later on used in the book [5]. An alternative approach to the 8V-model was developed in [3,4] where Baxter invented the *"eight-vertex" solid-on-solid* (SOS) model and solved it exactly by means of the co-ordinate Bethe ansatz. This approach provided another derivation of the same result (1.1)–(1.4), since the 8V-model is embedded within the SOS-model.

[1] Here we use the standard theta-functions [6], $\vartheta_i(u|\mathsf{q})$, $i=1,\ldots,4$, $\mathsf{q}=e^{\mathrm{i}\pi\tau}$, $\operatorname{Im}\tau>0$, with the periods π and $\pi\tau$. Our spectral parameter u is shifted with respect to that in [1] by a half of the imaginary period, see Section 2.5.1 for further details.