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Phase space structure of Chern–Simons theory with a non-standard puncture

C. Meusburger [a], B.J. Schroers [b,*]

[a] *Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, Ontario N2L 2Y5, Canada*
[b] *Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom*

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Abstract

We explicitly determine the symplectic structure on the phase space of Chern–Simons theory with gauge group $G \ltimes \mathfrak{g}^*$ on a three-manifold of topology $\mathbb{R} \times S^{\infty}_{g,n}$, where $S^{\infty}_{g,n}$ is a surface of genus g with $n+1$ punctures. At each puncture additional variables are introduced and coupled minimally to the Chern–Simons gauge field. The first n punctures are treated in the usual way and the additional variables lie on coadjoint orbits of $G \ltimes \mathfrak{g}^*$. The $(n+1)$st puncture plays a distinguished role and the associated variables lie in the cotangent bundle of $G \ltimes \mathfrak{g}^*$. This allows us to impose a curvature singularity for the Chern–Simons gauge field at the distinguished puncture with an arbitrary Lie algebra valued coefficient. The treatment of the distinguished puncture is motivated by the desire to construct a simple model for an open universe in the Chern–Simons formulation of $(2+1)$-dimensional gravity.

1. Introduction

Chern–Simons field theory has attracted the attention of both mathematicians and physicists. Its relevance in mathematics is largely due to its applications in fields such as knot theory, see [1] or the book [2], and the theory of moduli spaces of flat connections, see [3] for a summary. These research areas in turn provide useful concepts and methods for the study of Chern–Simons theory. From a physicist's point of view, Chern–Simons theory is interesting because it captures important aspects of real physical systems and is at the same time mathematically tractable. While not

* Corresponding author.
E-mail addresses: cmeusburger@perimeterinstitute.ca (C. Meusburger), bernd@ma.hw.ac.uk (B.J. Schroers).

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describing any real physical system, it plays an important role in, for example, the modelling of certain condensed matter systems. More importantly, it shares structural features with fundamental physical theories. Its diffeomorphism invariance, for instance, makes it a useful toy model for the investigation of coordinate independent approaches to quantisation. In particular, it is relevant to the study of Einstein's theory of gravity, which in $(2+1)$ dimensions can be formulated as a Chern–Simons theory [4,5].

The basic reason for the relative mathematical simplicity of Chern–Simons theory is that, with appropriate boundary conditions and after dividing out gauge degrees of freedom, it has a finite-dimensional phase space. Thus, in suitable circumstances Chern–Simons theory allows one to reduce the field-theoretic description of a physical system to a mathematically well-defined and finite-dimensional model. This reduction from an infinite to a finite number of degrees of freedom makes the classical theory tractable and considerably simplifies the quantisation.

In this paper we study Chern–Simons theory in its Hamiltonian formulation on a manifold of topology $\mathbb{R} \times S^{\infty}_{g,n}$, where $S^{\infty}_{g,n}$ is a surface of genus g with $n+1$ punctures. The purpose of the paper is to introduce a new way of treating punctures in Chern–Simons theory and to determine the physical phase space and its symplectic structure when one distinguished puncture is treated in this way. The main result is an explicit determination of the symplectic structure on the finite-dimensional physical phase space.

The motivation for our treatment of the distinguished puncture comes from the Chern–Simons formulation of $(2+1)$-dimensional gravity. As we explain in detail in a separate paper [6], the distinguished puncture can be used to model "spatial infinity" in open universes. Applied to $(2+1)$-dimensional gravity, our model leads to a finite-dimensional description of the phase space which can serve as a starting point for both an investigation of the classical dynamics and for quantisation, using the methods developed in [7]. The relation to $(2+1)$-dimensional gravity is also the motivation for our choice of gauge group. We consider gauge groups of the form $G \ltimes \mathfrak{g}^*$, where G is an arbitrary finite-dimensional Lie group. They include as special cases the Euclidean group and the Poincaré group in three dimensions which arise in the Chern–Simons formulation of Euclidean and Lorentzian $(2+1)$-gravity with vanishing cosmological constant [5]. Mathematically, groups of the form $G \ltimes \mathfrak{g}^*$ are particularly simple examples of Poisson–Lie groups, which is important in our analysis. We stress, however, that our treatment of the distinguished puncture and many of our results concerning the phase space are not limited to gauge groups of this type.

Since much of the paper is quite technical, we give a brief sketch of our treatment of punctures on $S^{\infty}_{g,n}$. The usual approach, followed in [1] and summarised in [3], is to require the curvature of the gauge field to have a delta-function singularity on the line $\mathbb{R} \times \{x_{(i)}\}$, where $x_{(i)}$ is the coordinate of the puncture on $S^{\infty}_{g,n}$, and to restrict the Lie algebra valued coefficient of the delta-function to a fixed coadjoint orbit of the gauge group. In order to achieve this, additional variables need to be introduced which parametrise the coadjoint orbit. The dynamics of these additional variables is governed by the Kirillov–Kostant–Souriau symplectic structure on the coadjoint orbit minimally coupled to the Chern–Simons gauge field. In this paper we treat the first n punctures on $S^{\infty}_{g,n}$ in this way. At the distinguished puncture, whose coordinate on $S^{\infty}_{g,n}$ is denoted x_∞, we also require the curvature to have a delta-function singularity, but this time we do not restrict the Lie algebra valued coefficient T. Instead we introduce an additional Lie group valued variable g at the distinguished puncture and interpret the pair (g, T) as an element of the cotangent bundle of the gauge group. The dynamics of the variables g and T is governed by the canonical symplectic structure on the cotangent bundle minimally coupled to the Chern–Simons gauge field.