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Eigenvalue amplitudes of the Potts model on a torus

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Abstract

We consider the Q-state Potts model in the random-cluster formulation, defined on *finite* two-dimensional lattices of size $L \times N$ with toroidal boundary conditions. Due to the non-locality of the clusters, the partition function $Z(L,N)$ cannot be written simply as a trace of the transfer matrix T_L. Using a combinatorial method, we establish the decomposition $Z(L,N) = \sum_{l,D_k} b^{(l,D_k)} K_{l,D_k}$, where the characters $K_{l,D_k} = \sum_i (\lambda_i)^N$ are simple traces. In this decomposition, the amplitudes $b^{(l,D_k)}$ of the eigenvalues λ_i of T_L are labelled by the number $l = 0, 1, \ldots, L$ of clusters which are non-contractible with respect to the transfer (N) direction, and a representation D_k of the cyclic group C_l. We obtain rigorously a general expression for $b^{(l,D_k)}$ in terms of the characters of C_l, and, using number theoretic results, show that it coincides with an expression previously obtained in the continuum limit by Read and Saleur.

1. Introduction

The Q-state Potts model on a graph $G = (V, E)$ with vertices V and edges E can be defined geometrically through the cluster expansion of the partition function [1]

$$Z = \sum_{E' \subseteq E} Q^{n(E')} \left(\mathrm{e}^J - 1\right)^{b(E')}, \tag{1.1}$$

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where $n(E')$ and $b(E') = |E'|$ are respectively the number of connected components (clusters) and the cardinality (number of links) of the edge subsets E'. We are interested in the case where G is a finite regular two-dimensional lattice of width L and length N, so that Z can be constructed by a transfer matrix T_L propagating in the N-direction.

In [2], we studied the case of cyclic boundary conditions (periodic in the N-direction and non-periodic in the L-direction). We decomposed Z into linear combinations of certain restricted partition functions (characters) K_l (with $l = 0, 1, \ldots, L$) in which l *bridges* (that is, marked non-contractible clusters) wound around the transfer (N) direction. We shall often refer to l as the *level*. Unlike Z itself, the K_l could be written as (restricted) traces of the transfer matrix, and hence be directly related to its eigenvalues. It was thus straightforward to deduce from this decomposition the amplitudes in Z of the eigenvalues of T_L. The goal of this work is to repeat this procedure in the case of toroidal boundary conditions.

Note that as in the cyclic case some other procedures exist. First, Read and Saleur have given in [3] a general formula for the amplitudes, based on the earlier Coulomb gas analysis of Di Francesco et al. [4]. They obtained that the amplitudes of the eigenvalues are simply $b^{(0)} = 1$ at the level $l = 0$ and $b^{(1)} = Q - 1$ at $l = 1$. For $l \geqslant 2$ they obtained that, contrary to the cyclic case, there are several differents amplitudes at each level l. Their number is equal to $q(l)$, the number of divisors of l. They are given by:

$$b^{(l,m)} = \Lambda(l, m; e_0) + (Q - 1)\Lambda\left(l, m; \frac{1}{2}\right), \tag{1.2}$$

where l is the level considered, and m is a divisor of l which labels the different amplitudes for a given level. Λ is defined as:

$$\Lambda(l, m; e_0) = 2 \sum_{d>0:\ d|l} \frac{\mu\left(\frac{m}{m\wedge d}\right)\phi\left(\frac{l}{d}\right)}{l\phi\left(\frac{m}{m\wedge d}\right)} \cos(2\pi d e_0). \tag{1.3}$$

Here, $m \wedge d$ denotes the greatest common divisor of m and d, and μ and ϕ are respectively the Möbius and Euler's totient function [5]. The Möbius function μ is defined by $\mu(n) = (-1)^r$, if n is an integer that is a product $n = \prod_{i=1}^r p_i$ of r *distinct* primes, $\mu(1) = 1$, and $\mu(x) = 0$ otherwise or if x is not an integer. Similarly, Euler's totient function ϕ is defined for positive integers n as the number of integers n' such that $1 \leqslant n' \leqslant n$ and $n \wedge n' = 1$. The value of e_0 depends on Q and is given by:

$$\sqrt{Q} = 2\cos(\pi e_0). \tag{1.4}$$

Note that in Eq. (1.3) we may write $\cos(2\pi d e_0) = T_{2d}(\sqrt{Q}/2)$, where $T_n(x)$ is the nth order Chebyshev polynomial of the first kind. The term $(Q - 1)\Lambda(l, m; \frac{1}{2})$ in Eq. (1.2) is due to configurations containing a cluster with "cross-topology" [3,4] (see later). The first few non-trivial amplitudes read explicitly:

$$\begin{aligned}
b^{(2,1)} &= \frac{1}{2}\left(Q^2 - 3Q\right), \\
b^{(2,2)} &= \frac{1}{2}\left(Q^2 - 3Q + 2\right), \\
b^{(3,1)} &= \frac{1}{3}\left(Q^3 - 6Q^2 + 8Q - 3\right), \\
b^{(3,3)} &= \frac{1}{3}\left(Q^3 - 6Q^2 + 8Q\right),
\end{aligned}$$