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Supersymmetric extensions of Schrödinger-invariance

Malte Henkel [a,b], Jérémie Unterberger [c]

[a] *Laboratoire de Physique des Matériaux,*[1] *Université Henri Poincaré Nancy I, B.P. 239, F-54506 Vandœuvre-lès-Nancy cedex, France*

[b] *Isaac Newton Institute for Mathematical Sciences, 20 Clarkson Road, Cambridge CB3 0EH, England*[2]

[c] *Institut Elie Cartan,*[3] *Université Henri Poincaré Nancy I, B.P. 239, F-54506 Vandœuvre-lès-Nancy cedex, France*

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Abstract

The set of dynamic symmetries of the scalar free Schrödinger equation in d space dimensions gives a realization of the Schrödinger algebra that may be extended into a representation of the conformal algebra in $d+2$ dimensions, which yields the set of dynamic symmetries of the same equation where the mass is not viewed as a constant, but as an additional coordinate. An analogous construction also holds for the spin-$\frac{1}{2}$ Lévy-Leblond equation. An $N=2$ supersymmetric extension of these equations leads, respectively, to a 'super-Schrödinger' model and to the (3|2)-supersymmetric model. Their dynamic supersymmetries form the Lie superalgebras $\mathfrak{osp}(2|2) \ltimes \mathfrak{sh}(2|2)$ and $\mathfrak{osp}(2|4)$, respectively. The Schrödinger algebra and its supersymmetric counterparts are found to be the largest finite-dimensional Lie subalgebras of a family of infinite-dimensional Lie superalgebras that are systematically constructed in a Poisson algebra setting, including the Schrödinger–Neveu–Schwarz algebra $\mathfrak{sns}^{(N)}$ with N supercharges. Covariant two-point functions of quasiprimary superfields are calculated for several subalgebras of $\mathfrak{osp}(2|4)$. If one includes both $N=2$ supercharges and time-inversions, then the sum of the scaling dimensions is restricted to a finite set of possible values.

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E-mail addresses: henkel@lpm.u-nancy.fr (M. Henkel), jeremie.unterberger@iecn.u-nancy.fr (J. Unterberger).

[1] Laboratoire associé au CNRS UMR 7556.
[2] Address until 30th of April 2006.
[3] Laboratoire associé au CNRS UMR 7502.

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1. Introduction

Symmetries have always played a central rôle in mathematics and physics. For example, it is well-known since the work of Lie (1881) that the free diffusion equation in one spatial dimension has a non-trivial symmetry group. It was recognized much later that the same group also appears to be the maximal dynamic invariance group of the free Schrödinger equation in d space dimensions, and it is therefore referred to as the *Schrödinger group* [1]. Its Lie algebra is denoted by $\mathfrak{sch}_d$. In the case $d=1$, one may realize $\mathfrak{sch}_1$ by the following differential operators

$$\begin{aligned}
&X_{-1} = -\partial_t, \qquad Y_{-1/2} = -\partial_r, && \text{time and space translations,}\\
&Y_{1/2} = -t\partial_r - \mathcal{M}r, && \text{Galilei transformation,}\\
&X_0 = -t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2}, && \text{dilatation,}\\
&X_1 = -t^2\partial_t - tr\partial_r - \frac{\mathcal{M}}{2}r^2 - 2xt, && \text{special transformation,}\\
&M_0 = -\mathcal{M}, && \text{phase shift.}
\end{aligned} \tag{1.1}$$

Here, $\mathcal{M}$ is a (real or complex) number and x is the scaling dimension of the wave function ϕ on which the generators of $\mathfrak{sch}_1$ act. The Lie algebra $\mathfrak{sch}_1$ realizes dynamical symmetries of the 1D free Schrödinger/diffusion equation $\mathcal{S}\phi = (2\mathcal{M}\partial_t - \partial_r^2)\phi = 0$ only if $x = 1/2$.

In particular, $\mathfrak{sch}_1$ is isomorphic to the semi-direct product $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathfrak{h}_1$, where $\mathfrak{sl}_2(\mathbb{R})$ is spanned by the three X-generators whereas the Heisenberg algebra in one space-dimension $\mathfrak{h}_1$ is spanned by $Y_{\pm 1/2}$ and M_0.

Schrödinger-invariance has been found in physically very different systems such as non-relativistic field-theory [2–4], celestial mechanics [5], the Eulerian equations of motion of a viscous fluid [6,7] or the slow dynamics of statistical systems far from equilibrium [8–10], just to mention a few. In this paper, we investigate the following two important features of Schrödinger-invariance in a supersymmetric setting. The consideration of supersymmetries in relation with Schrödinger-invariance may be motivated from the long-standing topic of supersymmetric quantum mechanics [11] and from the application of Schrödinger-invariance to the long-time behaviour of systems undergoing ageing, e.g. in the context of phase-ordering kinetics. Equations such as the Fokker–Planck or Kramers equations, which are habitually used to describe non-equilibrium statistical systems, are naturally supersymmetric, see [12,13] and references therein.

(1) First, there is a certain analogy between Schrödinger- and conformal-invariance. This is less surprising than it might appear at first sight since there is an embedding of the (complexified) Schrödinger Lie algebra in d space dimensions into the conformal algebra in $(d+2)$ space dimensions, $\mathfrak{sch}_d \subset (\mathfrak{conf}_{d+2})_{\mathbb{C}}$ [14,15].[4] This embedding comes out naturally when one thinks of the mass parameter $\mathcal{M}$ in the Schrödinger equation as an additional *coordinate*. Then a Laplace-transform of the Schrödinger equation with respect to $\mathcal{M}$ yields a Laplace-like equation which is known to be invariant under the conformal group.

(2) Second, we recall the fact, observed by one of us long ago [16], that the six-dimensional Lie algebra $\mathfrak{sch}_1$ can be embedded into the following infinite-dimensional Lie algebra with the

[4] In the literature, the invariance under the generator of special transformations X_1 is sometimes referred to as 'conformal invariance', but we stress that the embedding $\mathfrak{sch}_d \subset (\mathfrak{conf}_{d+2})_{\mathbb{C}}$ is considerably more general. In this paper, conformal invariance always means invariance under the whole conformal algebra $\mathfrak{conf}_{d+2}$.