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Recent progress in computing four-loop massive correlators *

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We report about recent progress in computing four-loop massive correlators. The expansion of these correlators in the external momentum leads to vacuum integrals. The calculation of these vacuum integrals can be used to determine Taylor expansion coefficients of the vacuum polarization function and decoupling functions in perturbative Quantum chromodynamics. New results at four-loop order for the lowest Taylor expansion coefficient of the vacuum polarization function and for the decoupling relation are presented.

1. Introduction

Two-point correlators have been studied in great detail in the framework of perturbative quantum field theory. Due to simple kinematics (only one external momentum) even multi-loop calculations can be performed. The results for all physically interesting diagonal and non-diagonal correlators and including *full* quark mass dependence are available up to $\mathcal{O}\left(\alpha_s^2\right)$ [1–3].
At four-loop order the two-point correlators can be considered in two limits. In the high energy limit massless propagators need to be calculated and in the low energy limit vacuum diagrams (tadpole integrals without dependence on the external momentum) arise. The evaluation of these massive tadpoles in three-loop approximation has been pioneered in ref. [4] and automated in ref. [5].
Similar to the three-loop case, the analytical evaluation of four-loop tadpole integrals is based on the traditional Integration-By-Parts (IBP) method. In contrast to the three-loop case the manual construction of algorithms to reduce arbitrary diagrams to a small set of master integrals is replaced by Laporta's algorithm [6, 7]. In this context the IBP identities are generated with numerical values for the powers of the propagators and the irreducible scalar products. In the next step, the resulting system of linear equations is then solved in the next step by expressing systematically complicated integrals in terms of simpler ones. The resulting solutions are then substituted into all the other equations.
This reduction has been implemented in an automated `FORM3` [8, 9] based program in which partially ideas described in ref. [7, 10, 11] have been implemented. The rational functions in the space-time dimension d, which arise in this procedure, are simplified with the program `FERMAT` [12]. The automated exploitation of *all* symmetries of the diagrams by reshuffling the powers of the propagators of a given topology in a unique way strongly reduces the number of equations which need to be solved.

In general, the tadpole diagrams encountered during our calculation contain both massive and massless lines. In contrast, the computation of the four-loop β-functions can be reduced to the evaluation of four-loop tadpoles composed of completely massive propagators. These special cases have been considered in [11, 13, 14].

The outline of this paper is as follows. In section 2 we discuss the calculation of the lowest expansion coefficient of the vacuum polarization function and present the results at four-loop order using methods as described above. In section 3 we present new results for the decoupling relation at four-loop order in perturbative QCD. Our

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conclusions are presented in section 4.

2. Vacuum polarization function

The vacuum polarization tensor $\Pi^{\mu\nu}(q)$ is defined as

$$\Pi^{\mu\nu}(q) = i \int dx\, e^{iqx} \langle 0|T j^\mu(x) j^\nu(0)|0\rangle\,, \qquad (1)$$

where q^μ is the external momentum and j^μ is the electromagnetic current of a heavy quark with mass m_h. The tensor $\Pi^{\mu\nu}(q)$ can be expressed by a scalar function, the vacuum polarization function $\Pi(q^2)$ through

$$\Pi^{\mu\nu}(q) = \left(q^\mu\, q^\nu - q^2\, g^{\mu\nu}\right)\, \Pi(q^2) + q^\mu\, q^\nu\, \Pi_L(q^2). \qquad (2)$$

The longitudinal part $\Pi_L(q^2)$ vanishes due to the Ward identity. The polarization function $\Pi(q^2)$ is related to the experimentally measurable R-ratio $R(s)$ through the dispersion relation:

$$\Pi(q^2) = \Pi(q^2=0) + \frac{q^2}{12\,\pi^2} \int ds \frac{R(s)}{s\,(s-q^2)}\,. \qquad (3)$$

Performing the n-th derivative of eq. (3) with respect to q^2 at $q^2 = 0$ one obtains the moments $\mathcal{M}_n^{\rm exp}$, which can be determined experimentally:

$$\mathcal{M}_n^{\rm exp} = \int ds \frac{R(s)}{s^{n+1}} = \frac{12\,\pi^2}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2)\bigg|_{q^2=0}\,.$$

The derivatives of the polarization function on the rhs are related to the Taylor expansion coefficients $\overline{C}_n$:

$$\overline{\Pi}(q^2) = \frac{3\,Q_q^2}{16\,\pi^2} \sum_{n\geq 0} \overline{C}_n\, z^n, \qquad (4)$$

($z = q^2/(4\,\overline{m}_h^2)$) which can be calculated in perturbative QCD. The first and higher derivatives are important for a precise determination of the charm- and bottom-quark mass (see e.g. [15]). But also the lowest expansion coefficient $\overline{C}_0$ has an interesting physical meaning: it relates the coupling of electromagnetic interaction in different renormalization schemes. In the case of QED-on-shell-renormalization the residue of the photon propagator is one and the electrical charge e coincides with the classical limit. If one performs renormalization in the $\overline{\rm MS}$-scheme one obtains a relation between the coupling constant of the electromagnetic interaction $\alpha_{\rm em} = e^2/(4\,\pi)$ in QED-on-shell-renormalization and the coupling constant $\overline{\alpha}_{\rm em} = \bar{e}^2/(4\,\pi)$ in the $\overline{\rm MS}$-scheme:

$$\alpha_{\rm em} = \frac{\overline{\alpha}_{\rm em}}{1 + \bar{e}^2\, \overline{\Pi}(q^2=0)}\,. \qquad (5)$$

For massive quarks, interacting through gluons, $\overline{\Pi}(q^2 = 0)$ has been evaluated in ref. [1]. At three-loop order in perturbative QCD this relation has already been determined in ref. [1]. For the QED case the corresponding result was calculated in ref. [4].

The first Taylor coefficient $\overline{C}_0$ has been calculated using the methods described in section 1. All tadpole diagrams were expressed through the set of 13 master integrals shown in figure 1. These master integrals have been calculated in refs. [16–18]. Inserting the master integrals into the lowest Taylor coefficient of the polarization

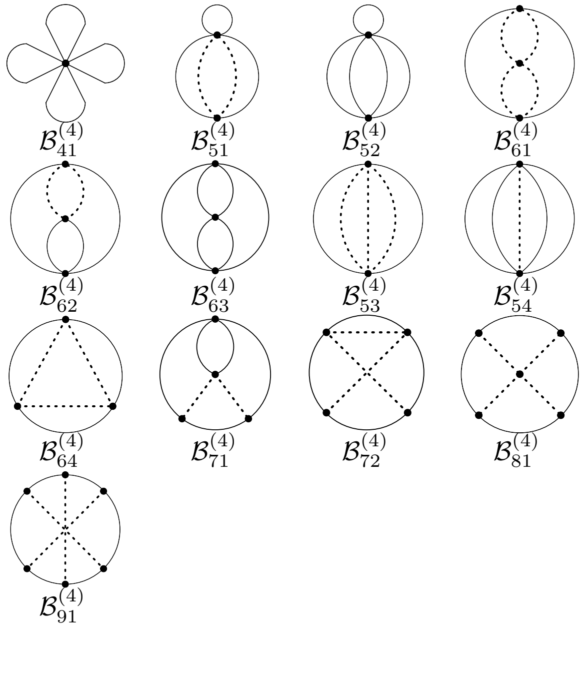

Figure 1. Master integrals. The solid lines denote massive lines, whereas the dashed lines denote massless lines.