Physics Letters B 760 (2016) 31–35

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Quantum Weyl invariance and cosmology

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ARTICLE INFO

Article history:
Received 5 March 2016
Received in revised form 12 June 2016
Accepted 16 June 2016
Available online 17 June 2016
Editor: N. Lambert

ABSTRACT

Equations for cosmological evolution are formulated in a Weyl invariant formalism to take into account possible Weyl anomalies. Near two dimensions, the renormalized cosmological term leads to a nonlocal energy-momentum tensor and a slowly decaying vacuum energy. A natural generalization to four dimensions implies a quantum modification of Einstein field equations at long distances. It offers a new perspective on time-dependence of couplings and naturalness with potentially far-reaching consequences for the cosmological constant problem, inflation, and dark energy.

To define a path integral over metrics in a quantum theory of gravity, one must introduce a regulator. Since the metric itself is a dynamical field, it is not clear in which metric to regularize and renormalize the theory, and how to ensure that the resulting answer is coordinate invariant and background independent. For this purpose it is convenient to enlarge the gauge symmetry to include Weyl invariance in addition to general coordinate invariance. This can be achieved by introducing a Weyl compensator field and a fiducial metric which scale appropriately keeping the physical metric Weyl invariant. The number of degrees of freedom remains the same upon imposing Weyl invariance. The path integral can now be regularized and renormalized using the fiducial metric.

A Weyl-invariant formulation has an important conceptual advantage because it separates scale transformations from coordinate transformations. The path integral can be regularized maintaining coordinate invariance at the quantum level. Weyl invariance can have potential anomalies in the renormalized theory but since it is a gauge symmetry all such anomalies must cancel. Coordinate invariance of the original theory then becomes equivalent to coordinate invariance plus *quantum* Weyl invariance of the modified theory. This procedure is well-studied in two dimensions where the Liouville field plays the role of the Weyl compensator and quantum Weyl invariance implies nontrivial scaling exponents.

There are both theoretical and phenomenological motivations to develop a Weyl-invariant formulation of gravity in higher dimensions, especially in the context of cosmology. Our chief theoretical motivation is to formulate the cosmological constant problem [1] in a manifestly gauge invariant way. The problem is usually stated in the language of effective field theories as a 'naturalness problem' analogous to the Higgs mass problem in electroweak theory or the strong-CP problem in quantum chromodynamics. The cosmological constant is the coupling constant of the identity operator added to the effective action. Since the identity has dimension zero, the cosmological constant term is the most relevant operator and should scale as M_0^d in d space-time dimension where the ultraviolet cutoff scale M_0 is at least of the order of a TeV. To reproduce the observed scale of the cosmological constant of the order of an meV, it is necessary to fine tune the bare vacuum energy.

This formulation of the cosmological constant problem is not entirely satisfactory. While the *generation* of the cosmological constant in the effective action depends only on short-distance physics, its *measurement* relies essentially on long-distance physics spanning almost the entire history of the universe. The physics of the cosmological constant thus spans more than a hundred logarithmic length scales. Moreover, all scales are evolving in a cosmological setting, and there is no preferred time for setting the cutoff in a manner that respects coordinate invariance. Thus, even to pose the cosmological constant problem properly, it is desirable to develop a formalism that accesses all time-scales in a gauge-invariant fashion.

A chief phenomenological motivation is to explore the possibility of effective time variation of vacuum energy. There is a substantial body of cosmological evidence for a slowly varying vacuum energy which is believed to have been responsible for an inflationary phase of exponential expansion in the very early universe. Observations of cosmic microwave background radiation indicate that the power spectrum generated during inflation is not strictly scale-free but has a slight red tilt. This implies that vacuum

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http://dx.doi.org/10.1016/j.physletb.2016.06.034

energy was not strictly constant but was slowly decaying during the inflationary era. Cosmological data also indicates that 69% of present energy density is in the form resembling vacuum energy. Time variation of dark energy is not established observationally at present but could be observed in planned observations. Any theoretical insight into the magnitude, equation of state, and time dependence of dark energy is clearly desirable.

Slowly varying vacuum energy can be represented by a cosmological constant Λ to first approximation. However, any time variation cannot be reintroduced simply by making Λ time-dependent because that would not be coordinate-invariant. A simple way to obtain time-dependent vacuum energy is to represent it by a slowly-rolling condensate of a scalar field. This idea is central to most current models of varying vacuum energy. Such a slowly-rolling field is called the 'inflaton' during the inflationary era and 'quintessence' during the present era. Models with scalar fields have the virtue of simplicity, but among the plethora of models none is particularly more compelling than the others; and our understanding of many questions of principle such as the initial value problem or the measure problem is less than satisfactory.

The puzzles regarding the cosmological constant, inflation, and dark energy all concern the nature of slowly varying vacuum energy. Occam's razor suggests that perhaps the essential underlying physics is governed by the same fundamental equations. With these motivations, I develop a Weyl-invariant formulation of quantum cosmology to explore the possibility of slowly evolving vacuum energy that does not rely on fundamental scalars.

I start with a Weyl-invariant reformulation of classical general relativity in d spacetime dimensions by introducing a Weyl compensator field Ω and a fiducial metric $h_{\mu\nu}$. Given a UV cutoff M_0, the reduced Planck scale M_p and the cosmological constant Λ correspond to dimensionless 'coupling constants' κ^2 and λ defined by:

$$M_p^{d-2} := \frac{M_0^{d-2}}{\kappa^2} \qquad \Lambda := \lambda\kappa^2 M_0^2 \,. \tag{1}$$

The gravitational action $I_K[h,\Omega]$ is given by

$$\frac{M_0^{d-2}}{2\kappa^2}\int dx\, e^{(d-2)\Omega}[R_h + (d-2)(d-1)|\nabla\Omega)|^2] \tag{2}$$

where all contractions are using the metric h and $dx := d^dx\sqrt{-h}$. The cosmological term is given by

$$I_\Lambda[h,\Omega] = -M_p^{d-2}\Lambda\int dx\, e^{d\Omega} = -\lambda M_0^d\int dx\, e^{d\Omega}\,. \tag{3}$$

All terms are coordinate invariant. Both I_K and I_Λ are separately invariant under Weyl transformations:

$$h_{\mu\nu} \to e^{2\xi}h_{\mu\nu}\,, \qquad \Omega \to \Omega - \xi\,. \tag{4}$$

Consequently both I_K and I_Λ satisfy the Ward identities for coordinate invariance:

$$\nabla^\nu\big(\frac{-2\,\delta I_a}{\sqrt{-h}\,\delta h^{\mu\nu}}\big) - \frac{1}{\sqrt{-h}}\frac{\delta I_a}{\delta\Omega}\nabla_\mu\Omega \equiv 0 \quad (a = K,\Lambda)\,, \tag{5}$$

and for Weyl invariance:

$$h^{\mu\nu}\big(\frac{-2\,\delta I_a}{\sqrt{-h}\,\delta h^{\mu\nu}}\big) - \frac{1}{\sqrt{-h}}\frac{\delta I_a}{\delta\Omega} \equiv 0 \quad (a = K,\Lambda)\,. \tag{6}$$

The physical metric $g_{\mu\nu} := e^{2\Omega}h_{\mu\nu}$ is Weyl invariant. In the 'physical' gauge we have $\Omega = 0$ and $h_{\mu\nu} = g_{\mu\nu}$ and (2) reduces to the Einstein–Hilbert action.

Consider a homogeneous and isotropic universe described by a spatially flat Robertson–Walker metric with scale factor $a(t)$, filled with a perfect fluid of energy density ρ and pressure p. The classical evolution of the universe is governed by the first Friedmann equation

$$H^2 = \frac{2\kappa^2\rho}{(d-2)(d-1)M_0^{d-2}} \tag{7}$$

and the conservation equation

$$\dot\rho = -(d-1)(p+\rho)H\,. \tag{8}$$

For a perfect fluid with a barotropic equation of state $p = w\rho$, the solutions to (7) and (8) are given by

$$\rho(t) = \rho_*(\frac{a}{a_*})^{-\gamma}\,, \qquad a(t) = a_*(1+\frac{\gamma}{2}H_*t)^{\frac{2}{\gamma}}\,, \tag{9}$$

where ρ_*, H_*, a_* are the initial values of various quantities at $t = 0$, and $\gamma := (d-1)(1+w)$. For the classical tensor of the cosmological term, $\rho_* = \lambda_* M_0^d$, $w = -1$, and $\gamma = 0$. As $\gamma \to 0$, the solution approaches nearly de Sitter spacetime with nearly exponential expansion and nearly constant density. Note that the cosmological evolution equations depend analytically on d, so one can 'analytically continue' the FLRW cosmologies.

Weyl invariance has potential anomalies at the quantum level. To gain intuition about these anomalies, we first consider spacetime near two dimensions, $d = 2+\epsilon$. To order ϵ, the total action I without matter is given by

$$\frac{q^2}{4\pi}\int dx\big(\frac{R_h}{\epsilon} + |\nabla\Omega|^2 + R_h\Omega - \frac{4\pi\lambda M_0^2}{q^2}e^{2\Omega}\big) \tag{10}$$

where the coupling constant q defined by

$$q^2 := \frac{2\pi\epsilon}{\kappa^2} \tag{11}$$

is held fixed as $\epsilon \to 0$. With $\chi := q\,\Omega$ and $\mu = \lambda M_0^2$, and ignoring the first term which depends only the fiducial metric, (10) is precisely the two-dimensional Liouville action with background charge q:

$$I[\chi] = \frac{1}{4\pi}\int dx\left(|\nabla\chi|^2 + q\,R_h\,\chi - 4\pi\,\mu\,e^{2\beta\chi}\right). \tag{12}$$

The field χ is sometimes called the 'timelike' Liouville field because the kinetic term has a wrong sign, as expected for the conformal factor of the metric. Classical Weyl invariance implies $\beta = 1/q$, but this relation receives quantum corrections because the operator $e^{2\beta\chi}$ is a composite operator with short-distance singularities. It can be renormalized treating χ as a free field [2] with the Green function G_2 of the Laplacian Δ_2:

$$\Delta_2^x\,G_2(x,y) = \delta_2(x,y)\,. \tag{13}$$

There is a short-distance divergence arising from self-contractions which combine into an exponential of the coincident Green function $G_2(x,x)$. This divergence can be regularized by using a heat kernel with a short-time cutoff. Renormalization then consists in subtracting a logarithmically divergent term from the regularized $G_2(x,x)$. This procedure is manifestly local and coordinate invariant. In two dimensions, any metric is conformal to the flat metric $\eta_{\mu\nu}$: $h_{\mu\nu} = e^{2\Sigma}\eta_{\mu\nu}$. The renormalized operator $\mathcal{O}_h(x) := [e^{2\beta\chi}]_h$ depends on the fiducial metric used for regularization and satisfies

$$\mathcal{O}_h(x) = e^{-2\beta^2\Sigma(x)}\,\mathcal{O}_\eta(x)\,. \tag{14}$$

The scalar Σ is a *nonlocal* functional of the metric:

$$\Sigma[h](x) := \frac{1}{2}\int dy\,G_2(x,y)R_h(y)\,. \tag{15}$$