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Newtonian, Post-Newtonian and Relativistic Cosmological Perturbation Theory

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Abstract

Newtonian cosmological perturbation equations valid to full nonlinear order are well known in the literature. Assuming the absence of the transverse-tracefree part of the metric, we present the general relativistic counterpart valid to full nonlinear order. The relativistic equations are presented without taking the slicing (temporal gauge) condition. The equations do have the proper Newtonian and first post-Newtonian limits. We also present the relativistic pressure correction terms in the Newtonian hydrodynamic equations.

1. Introduction

Cosmological perturbation theory is an important theoretical tool in interpreting cosmological observations like the two-dimensional temperature and polarization anisotropies of the cosmic microwave background radiation, the three-dimensional distribution and motions of galaxies, distorted images of galaxies due to gravitational lensing, etc. The cosmological perturbation equations are well known in the Newtonian context to fully nonlinear order [1], whereas the counterparts in Einstein's gravity are known in linear [2, 3] and low-order perturbation approximation [4]. Here, we present a self-contained summary of the basic equations of recently formulated fully nonlinear and exact cosmological perturbation theory in Einstein's gravity (Section 3). Comparisons are made with the Newtonian (Sections 2 and 4) and the post-Newtonian equations (Section 5). We also present the Newtonian equations in the presence of relativistic pressure (Section 6).

2. Newtonian Cosmological Perturbation Theory

Newtonian cosmological perturbation equations in the spatially homogeneous and isotropic background world model are [1]

$$\dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\widetilde{\varrho} = -\frac{1}{a}\nabla\cdot\left(\widetilde{\varrho}\mathbf{v}\right), \tag{1}$$

$$\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v} = \frac{1}{a}\nabla U - \frac{1}{a\widetilde{\varrho}}\nabla\widetilde{p}, \tag{2}$$

$$\frac{\Delta}{a^2}U = -4\pi G\left(\widetilde{\varrho} - \varrho\right). \tag{3}$$

These are the mass conservation, the momentum conservation, and the Poisson's equations, respectively; $\widetilde{\varrho}$, $\widetilde{p}$, $\mathbf{v}$, and U are the mass density, the pressure, the peculiar velocity, and the perturbed gravitational potential, respectively; $a(t)$ is the cosmic scale factor. We decompose the mass density and pressure to the background and perturbed parts as

$$\widetilde{\varrho} = \varrho + \delta\varrho, \quad \widetilde{p} = p + \delta p. \tag{4}$$

Evolution of the background world model is described by equation (21) properly derived in Einstein's gravity.

3. General Relativistic Cosmological Perturbation Theory

We consider the scalar- and vector-type perturbations in a *flat* background with the metric convention [3, 5]

$$ds^2 = -\left(1+2\alpha\right)c^2dt^2 - 2\chi_i c dt dx^i + a^2\left(1+2\varphi\right)\delta_{ij}dx^i dx^j, \tag{5}$$

where α, φ and χ_i are functions of spacetime with *arbitrary* amplitudes; index of χ_i is raised and lowered by δ_{ij} as the metric. We *ignored* the transverse-tracefree

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(TT) part of the metric which is interpreted as the gravitational waves to the linear perturbation order. The spatial part of the metric is simple because, in addition to ignoring the TT part, we already have taken the spatial gauge condition without losing any generality to the fully nonlinear order [3, 5].

We consider a fluid *without* anisotropic stress. The energy momentum tensor is given as

$$\widetilde{T}_{ab} = \widetilde{\varrho} c^2 \widetilde{u}_a \widetilde{u}_b + \widetilde{p} \left(\widetilde{g}_{ab} + \widetilde{u}_a \widetilde{u}_b \right), \qquad (6)$$

where tildes indicate the covariant quantities; $\widetilde{u}_a$ is the normalized fluid four-vector; $\widetilde{\varrho}$ includes the internal energy; in explicit presence of the internal energy we should replace

$$\widetilde{\varrho} \rightarrow \widetilde{\varrho} \left(1 + \frac{1}{c^2} \widetilde{\Pi} \right), \qquad (7)$$

where $\widetilde{\varrho}$ in the right-hand-side is the rest-mass density [6]. We introduce the following definitions of fluid three-velocities

$$\widetilde{u}_i \equiv a \frac{v_i}{c}, \quad \frac{1}{\widehat{\gamma}} v_i = \widehat{v}_i = \frac{1}{\mathcal{N}} \left[(1 + 2\varphi) \overline{v}_i - \frac{c}{a} \chi_i \right], \qquad (8)$$

where $\widehat{\gamma}$ is the Lorentz factor

$$\widehat{\gamma} \equiv \sqrt{1 + \frac{v^k v_k}{c^2 (1 + 2\varphi)}} = \frac{1}{\sqrt{1 - \frac{\widehat{v}^k \widehat{v}_k}{c^2 (1+2\varphi)}}} = \frac{1}{\sqrt{1 - \frac{1+2\varphi}{\mathcal{N}^2} \left(\frac{\overline{v}^k}{c} - \frac{\chi^k}{a(1+2\varphi)} \right) \left(\frac{\overline{v}_k}{c} - \frac{\chi_k}{a(1+2\varphi)} \right)}}, \qquad (9)$$

and $\mathcal{N}$ is related to the lapse function in (19). The velocities $\widehat{v}_i$ and $\overline{v}_i$ are more physically motivated ones [5]: $\widehat{v}_i$ is the fluid three-velocity measured by the Eulerian observer, and $\overline{v}_i$ is the coordinate three-velocity of fluid; the indices of v_i, $\widehat{v}_i$ and $\overline{v}_i$ are raised and lowered by δ_{ij}.

We can decompose χ_i and $\widehat{v}_i$ into the scalar- and vector-type perturbations to the nonlinear order as [5]

$$\chi_i = c\chi_{,i} + \chi_i^{(v)}, \quad \widehat{v}_i \equiv -\widehat{v}_{,i} + \widehat{v}_i^{(v)}, \qquad (10)$$

with $\chi^{(v)i}{}_{,i} \equiv 0 \equiv \widehat{v}^{(v)i}{}_{,i}$. We assign dimensions as

$$[a] = [\widetilde{g}_{ab}] = [\widetilde{u}_a] = [\alpha] = [\varphi] = [\chi^i] = [\widehat{v}^i/c] = 1,$$
$$[x^i] = L, \quad [\chi] = T, \quad [\widehat{v}/c] = L, \quad [\kappa] = T^{-1},$$
$$[\widetilde{T}_{ab}] = [\widetilde{\varrho} c^2] = [\widetilde{p}], \quad [G\widetilde{\varrho}] = T^{-2}, \qquad (11)$$

where κ, the perturbed part of the trace of extrinsic curvature, will be introduced below.

Here we present the complete set of fully nonlinear perturbation equations without taking the temporal gauge [5].

The definition of κ:

$$\kappa \equiv 3 \frac{\dot{a}}{a} \left(1 - \frac{1}{\mathcal{N}} \right) - \frac{1}{\mathcal{N}(1 + 2\varphi)} \left[3\dot{\varphi} + \frac{c}{a^2} \left(\chi^k{}_{,k} + \frac{\chi^k \varphi_{,k}}{1 + 2\varphi} \right) \right]. \qquad (12)$$

The ADM energy constraint:

$$-\frac{3}{2} \left(\frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3} \widetilde{\varrho} - \frac{\Lambda c^2}{3} \right) + \frac{\dot{a}}{a} \kappa + \frac{c^2 \Delta \varphi}{a^2 (1 + 2\varphi)^2} = \frac{1}{6} \kappa^2 - 4\pi G \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2} \right) \left(\widehat{\gamma}^2 - 1 \right) + \frac{3}{2} \frac{c^2 \varphi^{,i} \varphi_{,i}}{a^2 (1 + 2\varphi)^3} - \frac{c^2}{4} \overline{K}^i_j \overline{K}^j_i. \qquad (13)$$

The ADM momentum constraint:

$$\frac{2}{3} \kappa_{,i} + \frac{c}{2a^2 \mathcal{N}(1 + 2\varphi)} \left(\Delta \chi_i + \frac{1}{3} \chi^k{}_{,ik} \right) + 8\pi G \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2} \right) a \widehat{\gamma}^2 \frac{\widehat{v}_i}{c^2} = \frac{c}{a^2 \mathcal{N}(1 + 2\varphi)} \times \left\{ \left(\frac{\mathcal{N}_{,j}}{\mathcal{N}} - \frac{\varphi_{,j}}{1 + 2\varphi} \right) \left[\frac{1}{2} \left(\chi^j{}_{,i} + \chi_i{}^{,j} \right) - \frac{1}{3} \delta^j_i \chi^k{}_{,k} \right] - \frac{\varphi^{,j}}{(1 + 2\varphi)^2} \left(\chi_i \varphi_{,j} + \frac{1}{3} \chi_j \varphi_{,i} \right) + \frac{\mathcal{N}}{1 + 2\varphi} \nabla_j \left[\frac{1}{\mathcal{N}} \left(\chi^j \varphi_{,i} + \chi_i \varphi^{,j} - \frac{2}{3} \delta^j_i \chi^k \varphi_{,k} \right) \right] \right\}. \qquad (14)$$

The trace of ADM propagation:

$$-3 \frac{1}{\mathcal{N}} \left(\frac{\dot{a}}{a} \right)^{\cdot} - 3 \frac{\dot{a}^2}{a^2} - 4\pi G \left(\widetilde{\varrho} + 3 \frac{\widetilde{p}}{c^2} \right) + \Lambda c^2 + \frac{1}{\mathcal{N}} \dot{\kappa} + 2 \frac{\dot{a}}{a} \kappa + \frac{c^2 \Delta \mathcal{N}}{a^2 \mathcal{N}(1 + 2\varphi)} = \frac{1}{3} \kappa^2 + 8\pi G \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2} \right) \left(\widehat{\gamma}^2 - 1 \right) - \frac{c}{a^2 \mathcal{N}(1 + 2\varphi)} \left(\chi^i \kappa_{,i} + c \frac{\varphi^{,i} \mathcal{N}_{,i}}{1 + 2\varphi} \right) + c^2 \overline{K}^i_j \overline{K}^j_i. \qquad (15)$$

The tracefree ADM propagation:

$$\left(\frac{1}{\mathcal{N}} \frac{\partial}{\partial t} + 3 \frac{\dot{a}}{a} - \kappa + \frac{c \chi^k}{a^2 \mathcal{N}(1 + 2\varphi)} \nabla_k \right) \times \left\{ \frac{c}{a^2 \mathcal{N}(1 + 2\varphi)} \left[\frac{1}{2} \left(\chi^i{}_{,j} + \chi_j{}^{,i} \right) - \frac{1}{3} \delta^i_j \chi^k{}_{,k} - \frac{1}{1 + 2\varphi} \left(\chi^i \varphi_{,j} + \chi_j \varphi^{,i} - \frac{2}{3} \delta^i_j \chi^k \varphi_{,k} \right) \right] \right\} - \frac{c^2}{a^2 (1 + 2\varphi)} \left[\frac{1}{1 + 2\varphi} \left(\nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \varphi \right.$$