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# Total Born cross section of $e^+e^-$-pair production in relativistic ion collisions from differential equations

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ABSTRACT

We apply the differential equation method to the calculation of the total Born cross section of the process $Z_1Z_2 \to Z_1Z_2e^+e^-$. We obtain explicit expression for the cross section exact in the relative velocity of the nuclei.



## 1. Introduction

Theoretical investigation of electromagnetic $e^+e^-$ pair production in relativistic heavy-ion collisions goes back to the paper [1] where the Born cross section of the process at high energy was calculated in the leading logarithmic approximation. Racah, in his remarkable paper [2], has calculated the high-energy asymptotics of the Born cross section up to power-suppressed terms in $1/\gamma$ ($\gamma$ is a Lorentz factor of the colliding nuclei). Recently there was a certain rise of the interest to this process connected with the functioning of heavy ion colliders, like RHIC and LHC, see Ref. [3]. In particular, a great attention has been paid to the investigation of the Coulomb corrections to the cross section at high energies [4–8].

Speaking of the total Born cross section, the problem of its calculation is of a three-loop complexity level. Probably, this is the main reason why this quantity was not calculated exactly at arbitrary velocities of the colliding nuclei. This is in contrast to the Born cross section of pair photoproduction in the field of an ion, where the total Born cross section is known exactly for any energy of the initial photon since Refs. [9,10]. Now that we have an essential progress in the multiloop calculations, we are in position to fill this gap and to calculate the total Born cross section of $e^+e^-$ pair production in relativistic ion collisions.

The consideration of the present paper is based on the following approach. Using the optical theorem we express the total cross section via the sum of cut three-loop integrals. Then we apply the standard approach to multiloop calculations, based on the IBP reduction and differential equations for master integrals. The differential equations for the master integrals are first reduced to $\epsilon$-form [11] using the algorithm of Ref. [12], and then solved recursively up to the required order in $\epsilon$. Thus, we obtain the total Born cross section exactly in the relative velocity $\beta$ of the colliding nuclei. Our result perfectly agrees with the celebrated result of Racah [2] in the limit of large relativistic factor. At small $\beta$ we compare our result with estimate obtained in the recent paper [13] and find a complete disagreement. In order to find the origin of the disagreement, we perform a straightforward calculation of the low-energy asymptotics of the cross section differential with respect to the electron and positron momenta. The direct integration then reproduces our result obtained with the help of the differential equations.

## 2. Born cross section for the production of $e^+e^-$ pair

Using optical theorem, the total cross section of the process $Z_1Z_2 \to Z_1Z_2e^+e^-$ can be written as

$$\sigma = \frac{8\,\mathrm{Im}\,\mathcal{A}}{\gamma\beta}\,, \tag{1}$$

where $\mathrm{Im}\,\mathcal{A}$ is given by the sum of two cut diagrams depicted in Fig. 1, $\beta$ is the relative velocity of the colliding nuclei, and $\gamma = [1-\beta^2]^{-\frac{1}{2}}$ is the Lorentz factor. Contribution of both diagrams can be expresses in terms of the scalar integrals

$$I(n_1,\ldots,n_{12}) = \int \frac{d^dl\,d^dq_1\,d^dq_2}{(2\pi)^{3d}}\theta(q_1^0-l^0)\theta(q_2^0+l^0) \times \prod_{k=1}^{4}\mathrm{Im}\,\frac{1}{(D_k+i0)^{n_k}}\prod_{k=5}^{12}\frac{1}{(D_k+i0)^{n_k}}\,,$$

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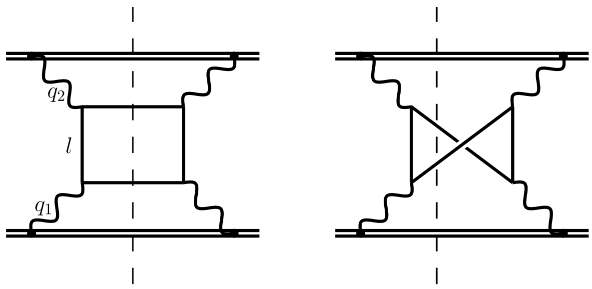


**Fig. 1.** Cut diagrams for the calculation of the total cross section of $e^+e^-$ pair production in the collisions of relativistic nuclei. Cut thin line denotes the cut propagator $-2\pi i\delta(p^2-m^2)(\hat{p}+m)$ of the electron, cut double line denotes the cut propagator $-2\pi i\delta(2u\cdot q)$ of a heavy particle, interaction vertex with the heavy particle is $-iu^\mu$ ($u=P/M$ is a four-velocity of the heavy particle).

$$
\begin{aligned}
&D_1=-2q_1\cdot u_1\,,\quad D_2=-2q_2\cdot u_2\,,\quad D_3=(l-q_1)^2-1\,,\\
&D_4=(l+q_2)^2-1\,,\quad D_5=l^2-1\,,\quad D_6=(l-q_1+q_2)^2-1\,,\\
&D_7=q_1^2\,,\quad D_8=q_2^2\,,\quad D_9=-2l\cdot u_1\,,\quad D_{10}=-2l\cdot u_2\,,\\
&D_{11}=-2q_2\cdot u_1\,,\quad D_{12}=-2q_1\cdot u_2\,. \qquad (2)
\end{aligned}
$$

Here $u_1$ and $u_2$ are the four-velocities of the nuclei, so that $u_1\cdot u_2=\gamma$.

We proceed in the following way. First, we perform the IBP reduction of the cut integrals from the above topologies in $d=4-2\epsilon$. For this step we use `LiteRed`, Refs. [14,15]. We end up with 8 master integrals

$$
\begin{aligned}
&J_1=I(1,1,1,1,0,0,0,1,0,0,0,0),\\
&J_2=I(1,1,1,1,0,1,0,0,0,0,0,0),\\
&J_3=I(1,1,1,1,0,2,0,0,0,0,0,0),\\
&J_4=I(1,1,1,1,-1,1,0,0,0,0,0,0),\\
&J_5=I(1,1,1,1,0,0,1,1,0,0,0,0),\\
&J_6=I(1,1,1,1,1,1,0,0,0,0,0,0),\\
&J_7=I(1,1,1,1,0,0,1,1,-1,-1,0,0),\\
&J_8=I(1,1,1,1,1,1,0,0,-1,0,-1,0)\,.
\end{aligned}
$$

Introducing the column-vector $\mathbf{J}=(J_1,\ldots,J_8)^T$, we obtain the differential system

$$\frac{\partial}{\partial\gamma}\mathbf{J}=M(\gamma,\epsilon)\mathbf{J}\,, \qquad (3)$$

where $M(\gamma,\epsilon)$ is a matrix with entries being rational functions of both $\gamma$ and $\epsilon$. Passing to new variable, $x=\frac{1-\beta}{1+\beta}$, we apply the algorithm from Ref. [12] to reduce the differential system (3) to $\epsilon$-form [11]. The differential system for the new basis $\widetilde{\mathbf{J}}=(\widetilde{J}_1,\ldots,\widetilde{J}_8)^T$ has the form

$$\frac{\partial}{\partial x}\widetilde{\mathbf{J}}=\epsilon\left[\frac{1}{x}M_0+\frac{1}{x-1}M_1+\frac{1}{x+1}M_2\right]\widetilde{\mathbf{J}}, \qquad (4)$$

$$M_0=\begin{bmatrix}-1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&-1&0&0&0&0&0\\0&0&0&-1&0&1&0&0\\0&0&0&0&3&3&0&0\\0&0&0&-1&-1&0&0&0\\0&0&0&0&0&0&-1&-1\\0&0&0&2&1&0&1&1\end{bmatrix}, \qquad (5)$$

$$M_1=\mathrm{diag}(2,0,2,2,-6,0,2,0)\,, \qquad (6)$$

$$M_2=\mathrm{diag}(0,0,0,0,0,0,0,-2)\,. \qquad (7)$$

We obtain $\epsilon$-expansion of $\widetilde{\mathbf{J}}=\sum_n\epsilon^n\widetilde{\mathbf{J}}^{(n)}$ term-by-term using the formula

$$\widetilde{\mathbf{J}}^{(n+1)}=\int dx\left[\frac{1}{x}M_0+\frac{1}{x-1}M_1+\frac{1}{x+1}M_2\right]\widetilde{\mathbf{J}}^{(n)}+\text{const} \qquad (8)$$

and fixing the constant from small-$\beta$ asymptotics. In order to calculate this asymptotics, we use the method of expansion by regions [16]. The only nontrivial boundary conditions come from $O(\beta^{2\epsilon-1})$ term in small-$\beta$ asymptotics of $J_1$ and $J_2$:

$$J_1\sim J_2\sim-\frac{2^{8\epsilon-16}\pi^{3\epsilon-5}\Gamma(\epsilon)^2\Gamma(2\epsilon-1)\Gamma(3\epsilon-2)}{\Gamma(4\epsilon-1)}\beta^{2\epsilon-1}\,. \qquad (9)$$

As a result, the $\epsilon$-expansions of both $\widetilde{\mathbf{J}}$ and $\mathbf{J}$ are expressed in terms of HPLs. Plugging the obtained expansions in the cross section expressed via $\widetilde{\mathbf{J}}$ we observe the cancellation of the terms $O(\epsilon^n)$ with $n=-4,\ldots,-1$. The $O(\epsilon^0)$ term gives us the result

$$
\begin{aligned}
\sigma=\frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\Bigg\{&-\frac{1-\beta^2}{12\beta^2}L^4+\frac{2\left(23\beta^2-37\right)S_{3a}}{9\beta^2}\\
&+\frac{2\left(11\beta^2-25\right)S_{3b}}{9\beta^2}-\frac{26S_2}{9\beta}\\
&-\frac{\left(\beta^6+217\beta^4-135\beta^2+45\right)L^2}{54\beta^6}\\
&+\frac{5\left(67\beta^4-48\beta^2+18\right)L}{27\beta^5}\\
&-\frac{2\left(78\beta^4-35\beta^2+15\right)}{9\beta^4}\Bigg\}, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
S_{3a}&=\mathrm{Li}_3\left(\frac{1-\beta}{1+\beta}\right)+L\,\mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)-\frac{L^2}{2}\log\left(\frac{2\beta}{1+\beta}\right)\\
&\quad-\frac{L^3}{12}-\zeta_3\,,\\
S_{3b}&=\mathrm{Li}_3\left(-\frac{1-\beta}{1+\beta}\right)+\frac{L}{2}\mathrm{Li}_2\left(-\frac{1-\beta}{1+\beta}\right)+\frac{L^3}{24}-\frac{\pi^2L}{24}+\frac{3\zeta_3}{4}\,,\\
S_2&=\mathrm{Li}_2\left(-\frac{1-\beta}{1+\beta}\right)+L\log\left(\frac{\beta+1}{2}\right)-\frac{L^2}{4}+\frac{\pi^2}{12}\,,\\
L&=\log\left(\frac{1+\beta}{1-\beta}\right).
\end{aligned}
$$

### *2.1. Asymptotics*

Given the expression (10), it is easy to calculate both high-energy and low-energy asymptotics of the total cross section. For $\gamma\gg1$ we have

$$
\begin{aligned}
\sigma=\frac{(Z_1\alpha)^2(Z_2\alpha)^2}{\pi m^2}\Bigg\{&\frac{28L_0^3}{27}-\frac{178L_0^2}{27}+\left(\frac{370}{27}+\frac{7\pi^2}{27}\right)L_0+\frac{7\zeta_3}{9}\\
&-\frac{13\pi^2}{54}-\frac{116}{9}-\frac{1}{\gamma^2}\bigg[\frac{4L_0^4}{3}-\frac{98L_0^3}{27}+\frac{188L_0^2}{27}\\
&-\left(\frac{172}{27}+\frac{25\pi^2}{54}\right)L_0-\frac{73\zeta_3}{18}+\frac{5\pi^2}{27}+\frac{43}{27}\bigg]+\ldots\Bigg\}, \qquad (11)
\end{aligned}
$$

where $L_0=\ln(2\gamma)$. The first line of Eq. (11) is the celebrated Racah's result [2], and the second line is the first correction to