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On ambiguity in knot polynomials for virtual knots

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ABSTRACT

We claim that HOMFLY polynomials for virtual knots, defined with the help of the matrix-model recursion relations, contain more parameters, than just the usual q and $A = q^N$. These parameters preserve topological invariance and do not show up in the case of ordinary (non-virtual) knots and links. They are most conveniently observed in the hypercube formalism: then they substitute q-dimensions of certain fat graphs, which are not constrained by recursion and can be chosen arbitrarily. The number of these new topological invariants seems to grow fast with the number of non-virtual crossings: 0, 1, 1, 5, 15, 91, 784, 9160, ... This number can be decreased by imposing the factorization requirement for composites, in addition to topological invariance – still freedom remains. None of these new parameters, however, appears in HOMFLY for Kishino unknot, which thus remains unseparated from the ordinary unknots even by this enriched set of knot invariants.

1. Introduction

The most efficient methods to calculate knot/link polynomials [1,2] are actually dealing with link diagrams – oriented graphs of valence $(2,2)$ with two types (colors) of vertices. In the Chern–Simons theory [3] this description appears in the temporal gauge $A_0 = 0$ [4] (while a choice of holomorphic gauge $A_1 + iA_2 = 0$ leads to the formalism of Kontsevich integrals [5]). Topological invariance is then provided by the Reidemeister invariance. For ordinary knots and links graphs are planar, non-planar graphs are interpreted as associated with *virtual* knots/links, provided the set of Reidemeister moves is appropriately enlarged [6]. Planar graphs are intimately related to braids, what allows to apply the powerful theory of Hecke algebras – which is in the base of the modern (universal) versions [7,8] of the Reshetikhin–Turaev (RT) formalism [9]. This is not the case for non-planar graphs, and these methods are not directly applicable (though available *answers* imply that just a minor – but still unknown – modification can be needed). Therefore virtual knot/link polynomials need to be studied by different methods. Historically first was the artful skein relation method [6] (which is normally a small part of the RT formalism, restricted to the fundamental representation [7]). A more systematic alternative approach is the hypercube method of [10,11], which was successfully applied to virtual knots in [12] and [13]. Moreover, in [13] it was reformulated in matrix model terms, what implies applicability of topological recursion ideas [14] – this seems to be the way, explaining emergence of skein relations in simple situations from the hypercube formalism. In fact the relations, found in [13] (which are probably equivalent to the older MOY relations [15]), are not quite recursive (what is also the case for the skein relations) – still, in combination with enumerative methods, they are sufficient to calculate fundamental HOMFLY for ordinary knots (this is currently checked up to 10 intersections – for the whole Rolfsen table). However, application of the same computer program (available at [16]) to virtual knots does *not* fix polynomials unambiguously. It is the purpose of this paper to describe this (empirical) situation in some detail.

In the formalism of [13] the basic objects are "quantum dimensions", associated with fat graphs, and above-mentioned relations are associated with graph reshufflings. What happens is that dimensions for some of the graphs remain undetermined – but this never

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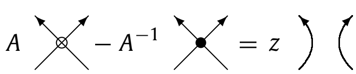
happens to graphs, appearing in consideration of ordinary (non-virtual) knots and links. A natural hypothesis is that we actually encounter *new* topological invariants, which are not observable in the world of ordinary knots. We did some checks ensuring that the new parameters are indeed topological invariants – but no proof is yet available.

One could suspect that such additional invariants exist – already because of the old puzzle of Kishino virtual knot [17,6], which is a composite of two unknots, not reducible to unknot by the Reidemeister–Kauffman moves. While Kishino knot polynomial does not depend on any of our new parameters – and thus remains undistinguished from unknot by fundamental HOMFLY – new parameters do appear for a closely related knot. This knot has the same shape of a planar diagram but all intersections black, so it is a composite of two virtual trefoils. Not only does its HOMFLY contain new parameters, it also contains *more* new parameters, then the virtual trefoil itself.

The paper is organized as follows. In Section 2 we remind how to calculate HOMFLY polynomial with help of skein relations. In Section 3 we show, that if we naively apply this method to virtual knots, we are faced with ambiguities. Then in Sections 4 and 5 we develop a proper language to address these issues – the hypercube formalism. In Section 6 we restate the ambiguity in this language, where it takes the form of new parameters – free values of some quantum dimensions. In Section 7 we describe an algorithm, that can check, to what extent keeping these parameters free is consistent with topological invariance. In Section 8 we present the outcome of this algorithm's work, so impatient reader may immediately look here. In Section 9 we study, whether the ambiguity can be constrained by imposing factorization of HOMFLY polynomial for composite knots. We finish the main body of the paper by formulating two *conjectures* about these new empirically observed free parameters. In Appendix A we consider in detail example of Kishino knot – showing that new parameters break factorization of composites. Appendix B contains more statistics on the results obtained by algorithm of Section 7. Finally, in Appendix C we show an alternative line of development of the hypercube formalism – using symmetric representation [2] instead of antisymmetric [1, 1].

2. Skein method = skein relations + Reidemeister moves

One of the ways to calculate fundamental HOMFLY polynomial for a given planar diagram of a knot is to use the skein relation



(1)

where $A = q^N$ and $z = q - q^{-1}$. From the point of view of RT approach this relation is the property of quantum $\mathcal{R}$-matrix in the fundamental representation

$$\left(\mathcal{R} - \frac{1}{Aq}\right)\left(\mathcal{R} + \frac{q}{A}\right) = 0 \tag{2}$$

(for higher representations $\mathcal{R}$-matrix has more different eigenvalues, and the story gets more involved).

The method is to gradually express HOMFLY polynomial for a diagram through HOMFLY polynomials of simpler diagrams, eventually expressing everything though completely unknotted diagrams.

Each simplification step consists of two substeps:

- skein relation is applied to some crossings of a planar diagram
- some of resulting summands are simplified with the help of Reidemeister moves

Crucial is the second substep, without it the procedure would not terminate at all, because one of the summands after the application of relation always has *the same* number of crossings as in the original diagram. But can we always simplify this summand with help of Reidemeister moves, such that it contains less crossings, than the diagram we started with?

The answer is yes – if we choose crossings, at which skein relations are applied, in a clever way. Namely, for non-virtual knots it is easy to observe that

Proposition 2.1. *Every planar diagram can be unknotted by changing some of its crossings for inverse ones.*

Thus, if we choose to apply skein relation precisely at all these "unknotting" crossings, then the problematic summand with non-decreased number of crossings can definitely be simplified – even completely unknotted! – with the help of Reidemeister moves – because it *is* the unknot. Thus *in principle* the procedure *can* calculate arbitrary fundamental HOMFLY.

The problem in practice is to recognize, which crossings are the unknotting ones? If we are not concerned with efficiency and just want to show that algorithm terminates in finite time, we can choose the most naive, brute-force enumerative approach.

We just loop over all $2^{\#\text{crossings}}$ possible choices and check, whether diagram we obtain by changing these crossings is an unknot. To check, whether some diagram is an unknot is a famous "unknotting problem". Again we can choose the most naive brute-force approach to it.

The number of Reidemeister moves, needed to unknot any planar diagram, is at most exponential in the number of crossings [20]. Hence, the full search over all possible sequences of Reidemeister moves (of restricted length) will take at most doubly exponential time – but *finite* nonetheless. Hence, *skein method will give an answer for any planar diagram in finite time as well.*

Of course, in practice, it is possible to optimize this brute-force procedure in many ways. First of all, it is very often possible to apply skein relation at just one crossing, but in such a way, that the number of crossings of the problematic summand can be immediately reduced with help of the second Reidemeister move. Second, there are a lot of smarter approaches to the unknotting problem, which are at most exponential in the number of crossings (though it is still unknown, whether there is a polynomial algorithm for that).