Topological index theorem on the lattice through the spectral flow of staggered fermions

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ABSTRACT

We investigate numerically the spectral flow introduced by Adams for the staggered Dirac operator on realistic (quenched) gauge configurations. We obtain clear numerical evidence that the definition works as expected: there is a clear separation between crossings near and far away from the origin, and the topological charge defined through the crossings near the origin agrees, for most configurations, with the one defined through the near-zero modes of large taste-singlet chirality of the staggered Dirac operator. The crossings are much closer to the origin if we improve the Dirac operator used in the definition, and they move towards the origin as we decrease the lattice spacing.

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1. Introduction

As is well known, smooth SU(N) gauge fields in a 4-dimensional compact differentiable manifold M have associated an integer topological charge

$$Q = -\frac{1}{32\pi^2} \int_M d^4x \, \epsilon_{\mu\nu\rho\sigma} \, \text{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right] \tag{1}$$

where $F_{\mu\nu}$ is the gauge potential.

This is not merely a mathematical curiosity; it plays a fundamental role in the understanding of the $U_A(1)$ problem through the anomaly [1–3], and the Witten–Veneziano formula [4,5]. It is also crucial for the investigation of the θ vacuum in QCD and the strong CP problem [6], and therefore with the current experimental searches for axions [7,8].

The issue of obtaining a theoretically sound and practical definition for the topological charge of lattice gauge fields is an old one. Several definitions exist, each with its own advantages and disadvantages. Some such definitions are purely gluonic, essentially transcribing the continuum definition to the lattice, whereas others

take advantage of the index theorem and compute the charge as the index of a conveniently chosen fermionic operator.

In [9], Adams introduced a new definition of the index of a staggered Dirac operator, based on the spectral flow of a related hermitian operator. Some numerical results were obtained there for synthetic configurations in the *2D* U(1) model.

The purpose of this paper is to study systematically Adams' definition in 4D (quenched) QCD (preliminary results were presented in [10]).

2. Definition of the topological charge

The basic observation of Adams in [9] is that when considering the spectral flow definition of the topological charge in the continuum, there is some freedom in the choice of the relevant hermitian operator. By a suitable choice, we can construct an operator which, when implemented in the lattice with a staggered Dirac operator, has all the required properties.

In the continuum, for a given gauge field, one usually considers the spectral flow of the hermitian operator

$$H(m) = \gamma_5 \, (D - m) \tag{2}$$

as a function of m. Because of the key property that

$$H(m)^2 = D^\dagger D + m^2, \tag{3}$$

if we trace the flow of eigenvalues $\{\lambda(m)\}$ of H, the ones corresponding to the zero modes of D, and only those, will change sign at the origin $m = 0$, each with a slope ± 1 which depends on the

chirality of the corresponding mode. This gives us the index of D, and through the index theorem, the topological charge Q of the corresponding gauge configuration.

On the lattice, we can substitute in (2) D by the discretized Wilson Dirac operator

$$H_W(m) = \gamma_5 (D_W - m). \tag{4}$$

Now the index can be obtained similarly to the continuum, by counting the number of eigenvalues of $H(m)$ that change sign close to the origin $m = 0$, taking into account the slope of such crossings. If we try to do the same with the lattice staggered Dirac operator D_{st}, we realize that the procedure does not work anymore, as the corresponding H fails to be hermitian.

The key innovation in Adams [9] is the realization that one can use a different H in the continuum to accomplish the same task, namely

$$H(m) = iD - m\gamma_5. \tag{5}$$

This operator is hermitian and verifies (3), and therefore its spectral flow also gives the index of D. But now we can substitute in (5) D by the lattice staggered discretization,

$$H_{st}(m) = iD_{st} - m\Gamma_5, \tag{6}$$

where D_{st} is the massless staggered Dirac operator and Γ_5 is the taste-singlet staggered γ_5 [11]. This operator is hermitian, and we can study its spectral flow, $\lambda(m)$. The would-be zero modes of D_{st} are identified with the eigenmodes for which the corresponding eigenvalue flow $\lambda(m)$ crosses zero at low values of m, and the chirality of any such mode equals (with our conventions) the sign of the slope of the crossing [9].

Any staggered discretization of the Dirac operator can be used, in principle, to implement (6). We have chosen to work with the unimproved, 1-link staggered Dirac operator [12], and with the highly improved HISQ discretization [13]. In each case we have calculated, using standard numerical algorithms, the smallest (in absolute value) 20 eigenvalues of $H_{st}(m)$ for enough values of m to allow us to determine unambiguously the cuts with the x axis. To compare with previous work, we have also calculated the low-lying modes of the HISQ Dirac operator at $m = 0$, and identify the would-be zero modes with the high taste-singlet chirality ones [14,15].

3. 2D U(1)

We started by studying the behavior of the spectral flow of $H_{st}(m)$ corresponding to the 1-link staggered Dirac operator in 2D U(1) lattice gauge theory; our purpose was twofold: testing our numerical framework in a less demanding setting, and working in a theory with a simple geometrical definition of the topological charge (even if not immune from problems arising from dislocations [16]). To this end we can either construct gauge fields configurations with an assigned topological charge [17], or generate realistic field configurations from a canonical ensemble, selecting those with the charge Q we are interested in. We used the first possibility in the earlier stage of our tests and then we moved to the other option: the results shown in what follows are from this second phase.[2]

We have generated a large ensemble of quenched configurations at different values of β, varying from 4 to 9, in order to cover the scaling region for the lattice size we consider (L = 60). Then we
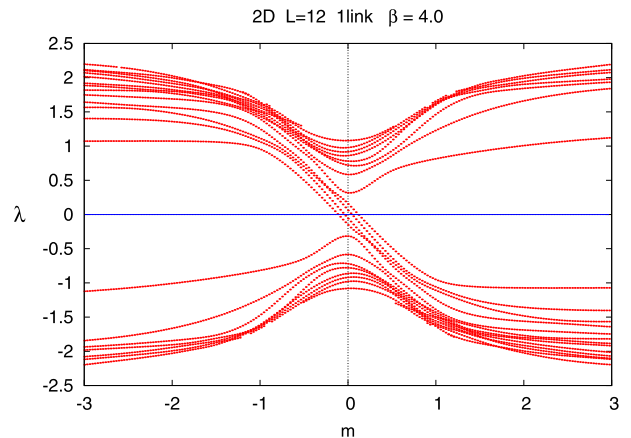

2D L=12 1link β = 4.0

Fig. 1. Spectral flow of the 2D 1link Dirac operator for a gauge configuration with $Q = -2$.

select among this set of configurations subsets of fixed charge Q, for which we computed the spectral flow.

The spectrum of (6) has the exact symmetry $\lambda(m) \leftrightarrow -\lambda(-m)$, therefore we only need to calculate the flow for, say, $m > 0$. An equal number of crossings, with identical slopes, will be present for $m < 0$.

In Fig. 1 we show the spectral flow for a gauge configuration with a charge $Q = -2$, corresponding to a 12^2 lattice and a coupling constant $\beta = 4.0$. We plot the lowest (in absolute value) 20 eigenvalues of $H(m)$ in the range $(-3, 3)$. We can appreciate two crossings with negative slopes for $m < 0$, and the symmetric ones for $m > 0$.[3]

In Fig. 2 we show the spectral flows corresponding to several gauge configurations in a larger volume, 60^2, and at several values of the coupling. From now on we only plot the flow for $m > 0$. For clarity, in most of the figures we plot the variable

$$\tilde{\lambda} = \text{sgn}(\lambda)\sqrt{|\lambda|}\log(|\lambda|) \tag{7}$$

versus $\log(m)$. As we can see in every case we have the expected number of crossings.

4. 4D quenched QCD

For our numerical calculations in 4D we have used three ensembles of tree-level Symanzik and tadpole improved quenched QCD at three values of the coupling constant β (5.0, 4.8 and 4.6), corresponding respectively to lattice spacings of approximately 0.077, 0.093 and 0.125 fm [15], with lattice volumes of 20^4, 16^4 and 12^4 respectively. These ensembles are thus approximately matched in physical volume, ≈ 1.5 fm^4.

In Figs. 3 and 4 we show the spectral flow for several representative configurations from the coarsest and finest ensembles, and in each case for H_{st} built from both the 1-link and the HISQ Dirac operators. As before, we calculate the first 20 eigenvalues in absolute value. The topological charges were also calculated by counting the number of eigenvectors of the HISQ Dirac operator with high chirality [15], and in each case there is agreement between the two definitions and for the spectral flow corresponding to both operators. We can appreciate in the figures that the cuts of the spectral flow with the x axis are closer to the origin $m = 0$

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[2] See [18] for related work in the unquenched case.

[3] Staggered fermions in D dimensions have a taste degeneracy of $2^{D/2}$, and therefore the number of crossings in the continuum gets multiplied by that factor in the lattice.