Justifying the naive partonic sum rule for proton spin

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A B S T R A C T

We provide a theoretical basis for understanding the spin structure of the proton in terms of the spin and orbital angular momenta of free quarks and gluons in Feynman's parton picture. We show that each term in the Jaffe–Manohar spin sum rule can be related to the matrix element of a gauge-invariant, but frame-dependent operator through a matching formula in large-momentum effective field theory. We present all the matching conditions for the spin content at one-loop order in perturbation theory, which provide a basis to calculate parton orbital angular momentum in lattice QCD at leading logarithmic accuracy.
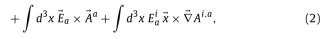
Understanding the spin structure of the proton has been an important goal in hadron physics. In the past 25 years, two well-known spin sum rules have been proposed to analyze the content of the proton spin. The first, proposed by Jaffe and Manohar [1], was motivated from a free-field expression of QCD angular momentum boosted to the infinite momentum frame (IMF) of the proton. The second is the frame-independent and manifestly gauge-invariant decomposition by one of the authors [2]. Notwithstanding that the latter has received considerable attention for its relation to generalized parton distributions (GPDs) and experimental probes [2–4], it is not natural in the language of parton physics (see, however, a recent discussion on its connection to the transverse polarization [5]). In contrast, the Jaffe–Manohar sum rule defined in the light-cone gauge $A^+ = 0$ has a natural partonic interpretation, as the proton spin can be decomposed into four parts,

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma(\mu) + \Delta L_q(\mu) + \Delta G(\mu) + \Delta L_g(\mu), \tag{1}$$

where the individual terms are the spin and orbital angular momenta (OAM) of the quark and gluon partons, respectively, and μ is the renormalization scale. All the four terms are defined to be the proton matrix elements of free-field angular momentum operators (AMOs) in IMF [1]:

$$\vec{J} = \int d^3x\, \psi^\dagger \frac{\vec{\Sigma}}{2} \psi + \int d^3x\, \psi^\dagger \vec{x} \times (-i\vec{\nabla})\psi$$

$$+ \int d^3x\, \vec{E}_a \times \vec{A}^a + \int d^3x\, E_a^i\, \vec{x} \times \vec{\nabla} A^{i,a}, \tag{2}$$

where $E^i = F^{i+}$, a and i are the color and spatial indices. However, the free-field form of the angular momentum in gauge theories faces a conceptual problem: all terms except the first one are gauge dependent, and it is unclear why the light-cone gauge operator is measurable in physical experiments.

In the past two decades, there has been a long list of literatures attempting to justify the Jaffe–Manohar sum rule as physical (see e.g., [6–12]). There are strong motivations behind this: First, ΔG as defined in the light-cone gauge is measurable in high-energy experiments, although this appears to be a theoretical puzzle by itself—while ΔG is easy to define from the Feynman parton picture, there is no natural gauge-invariant notion for the spin of gauge particles [13]. Second, when the proton is probed in IMF, some of its physical properties can be understood from simple addition of those of free quarks and gluons. For example, the longitudinal momentum of the proton is the sum of that of the quark and gluon partons:

$$1 = \int dx\, x \left(\sum_q q(x) + g(x) \right), \tag{3}$$

where $q(x)$ and $g(x)$ are the unpolarized quark and gluon momentum distribution functions. This simple parton picture may work for the proton spin as well.

It was first proposed that although the free-field AMOs are gauge dependent, their physical matrix elements are gauge invariant [7]. A similar claim was also made recently [8]. However, this is

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invalidated by a one-loop calculation by the present authors [10], where the matrix element of the gluon spin operator was shown to be different in the Coulomb and light-cone gauges [14] (for more general discussions see Ref. [9]). Actually, as argued in Refs. [11,10], for the bound-state proton, there is no physically meaningful notion of gluon spin or OAM due to the existence of longitudinal gluons. Only when the proton is boosted to IMF, the longitudinal component of gluons is suppressed by the infinite boost and the gluons can be regarded as free radiation. This is the well-known Weizsäcker–Williams (WW) approximation [15]. The gluon spin then acquires a clear physical meaning and can be represented by $\vec{E} \times \vec{A}$, but is subject to a class of "physical" gauge conditions that leave the transverse polarizations of the gluon field intact [12]. Similar arguments also apply to the quark and gluon OAM. Therefore, we can regard the free-field form in the Jaffe–Manohar sum rule as physical if we work in IMF with a "physical" gauge condition. This is equivalent to using the light-cone coordinates and gauge [1], and the reason is simple: All the "physical" gauges will flow into the light-cone gauge in the IMF limit [12].

From a practical perspective, the Jaffe–Manohar sum rule still poses difficulty for a nonperturbative lattice calculation of its individual contributions, because the explicit usage of light-cone coordinates and gauge brings real-time dependence. One may avoid this difficulty by using normal space–time coordinates with a "physical" gauge that does not involve time, and calculating with a proton at infinite momentum. However, the largest momentum attainable on the lattice with spacing a is constrained by the lattice cutoff π/a.

The above difficulty can, however, be circumvented in the framework of *large-momentum effective field theory* (LaMET) [16] proposed by one of the authors. Suppose one is to calculate some light-cone or parton observable $\mathcal{O}$. Instead of computing it directly, one defines, in the LaMET framework, a quasi-observable $\tilde{\mathcal{O}}$ that depends on a large hadron momentum P^z. In general, both $\mathcal{O}$ and $\tilde{\mathcal{O}}$ suffer from ultraviolet (UV) divergences. If $P^z \to \infty$ is taken prior to a UV regularization, the quasi-observable $\tilde{\mathcal{O}}$ becomes the parton observable $\mathcal{O}$ by construction. However, what one can calculate in practice is the quasi-observable $\tilde{\mathcal{O}}$ at large but finite P^z with UV regularization imposed first. This is the case in lattice computations. The difference between $\mathcal{O}$ and $\tilde{\mathcal{O}}$ is just the order of limits. This is similar to an effective field theory set-up. The difference is that here the role of perturbative degrees of freedom is played by the large momentum of the external state, hence it cannot be arranged into a Lagrangian formalism. Nevertheless, one can bridge the quasi- and parton observables through

$$\tilde{\mathcal{O}}(P^z/\Lambda) = Z\left(P^z/\Lambda, \mu/\Lambda\right)\mathcal{O}(\mu) + \frac{c_2}{(P^z)^2} + \frac{c_4}{(P^z)^4} + \cdots, \quad (4)$$

where Λ is a UV cutoff imposed on the quasi-observable, and c_i's are higher-twist contributions suppressed by powers of P^z. That is, the quasi-observable $\tilde{\mathcal{O}}(P^z/\Lambda)$ can be factorized into the parton observable $\mathcal{O}(\mu)$ and a matching coefficient Z, up to power suppressed corrections. Taking the $P^z \to \infty$ limit does not change the infrared (IR) behavior of the observable, but only affects its UV behavior. Therefore $\mathcal{O}(\mu)$ captures all the IR physics in $\tilde{\mathcal{O}}(P^z/\Lambda)$, and the matching coefficient Z is completely perturbative.

An explicit example of Eq. (4) is presented in Refs. [17,18] for the case of parton distribution functions (PDFs), where the factorization formula has a convolution form, and the matching coefficients were calculated at the leading logarithmic order. Using these results, the first direct lattice calculation of the isovector sea-quark parton distributions has been available recently [19]. A similar factorization formula was also proposed in Ref. [20] to extract PDFs from lattice QCD calculations based on QCD factorization of lattice "cross sections".

Within the LaMET framework, we can start with suitable quasi-observables to calculate the proton spin content. According to our discussions above, these quasi-observables can be defined as the free-field QCD AMOs in a "physical" gauge condition that has the correct WW approximation in the IMF limit [12]. A possible choice of the "physical" gauge condition is the expression in terms of nonlocal operators introduced by Chen et al. [21,22]:

$$\vec{J}_{QCD} = \int d^3x\, \psi^\dagger \frac{\vec{\Sigma}}{2}\psi + \int d^3x\, \psi^\dagger \vec{x} \times (-i\vec{\nabla} - g\vec{A}_\parallel)\psi$$
$$+ \int d^3x\, \vec{E}_a \times \vec{A}_\perp^a + \int d^3x\, E_a^i\,(\vec{x} \times \vec{\nabla})A_\perp^{i,a}, \quad (5)$$

where $\vec{x}$ are the spatial coordinates, and $\vec{A}$ is decomposed into a pure-gauge part $\vec{A}_\parallel$ and a physical part $\vec{A}_\perp$ which satisfy (see also Ref. [23])

$$\partial^i A_\parallel^{j,a} - \partial^j A_\parallel^{i,a} - gf^{abc}A_\parallel^{i,b}A_\parallel^{j,c} = 0,$$
$$\partial^i A_\perp^i - ig[A^i, A_\perp^i] = 0, \quad (6)$$

so that each term in Eq. (5) is gauge invariant. From Eq. (6), one can show that in the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, $\vec{A}_\perp$ equals $\vec{A}$ order by order in perturbation theory. Therefore, Eq. (5) corresponds to choosing the Coulomb gauge as the "physical" gauge.

It has been shown in Ref. [10] that $\vec{E}_a \times \vec{A}_\perp^a$ in Eq. (5) is equivalent to the total gluon spin operator in the IMF limit. It is easy to see that the other nonlocal terms in Eq. (5) also have the correct WW approximation as the parton OAM. Therefore, we can choose the nonlocal operators in Eq. (5) as the quasi-observables for parton angular momentum in the LaMET approach.

The advantage of the expression in Eq. (5) is that it is time independent and thus allows for a direct calculation in lattice QCD. Suppose we evaluate the matrix elements of these quasi-observables with finite momentum P^z, we should have

$$\frac{1}{2} = \frac{1}{2}\Delta\tilde{\Sigma}(\mu, P^z) + \Delta\tilde{G}(\mu, P^z)$$
$$+ \Delta\tilde{L}_q(\mu, P^z) + \Delta\tilde{L}_g(\mu, P^z), \quad (7)$$

where the P^z-dependence is expected since Eq. (5) is a frame-dependent expression [10]. Following the effective theory argument above, we can relate these quasi-observables to the corresponding parton observables through the following factorization formula:

$$\Delta\tilde{\Sigma}(\mu, P^z) = \Delta\Sigma(\mu),$$
$$\Delta\tilde{G}(\mu, P^z) = z_{qg}\Delta\Sigma(\mu) + z_{gg}\Delta G(\mu) + O\left(\frac{M^2}{(P^z)^2}\right),$$
$$\Delta\tilde{L}_q(\mu, P^z) = P_{qq}\Delta L_q(\mu) + P_{gq}\Delta L_g(\mu)$$
$$+ p_{qq}\Delta\Sigma(\mu) + p_{gq}\Delta G(\mu) + O\left(\frac{M^2}{(P^z)^2}\right),$$
$$\Delta\tilde{L}_g(\mu, P^z) = P_{qg}\Delta L_q(\mu) + P_{gg}\Delta L_g(\mu)$$
$$+ p_{qg}\Delta\Sigma(\mu) + p_{gg}\Delta G(\mu) + O\left(\frac{M^2}{(P^z)^2}\right), \quad (8)$$

where M is the proton mass, and all the matrix elements are renormalized in the $\overline{MS}$ scheme. $\Delta\tilde{\Sigma}(\mu, P^z)$ is the same as $\Delta\Sigma(\mu)$ because the quark spin operator is frame independent and should have the same matrix elements in the Coulomb and light-cone gauges. The z_{ij}, P_{ij} and p_{ij}'s are the matching coefficients to be calculated in perturbative QCD.

In the remainder of this paper, we show how to obtain all the matching coefficients in Eq. (8) at one-loop order. First, let us take