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Reversing the critical Casimir force by shape deformation

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ABSTRACT

The exact critical Casimir force between periodically deformed boundaries of a 2D semi-infinite strip is obtained for conformally invariant classical systems. Only two parameters (conformal charge, dimension of a boundary changing operator), along with the solution of an electrostatic problem, determine the Casimir force, rendering the theory practically applicable to any shape. The attraction between any two mirror symmetric objects follows directly from our general result. The possibility of purely shape induced reversal of the force, as well as occurrence of stable equilibrium is demonstrated for certain conformally invariant models, including the tricritical Ising model.

Fluctuation-induced forces (FIF) are ubiquitous in nature [1]; prominent examples include van der Waals [2], and closely related Casimir forces [3,4], originating from quantum fluctuations of the electromagnetic field. Thermal fluctuations in soft matter also lead to FIF, most pronounced near a critical point where correlation lengths are large [5,6]. Controlling the sign of FIF (attractive or repulsive) is important to myriad applications in design and manipulation of micron scale devices. Theoretical results for FIF in various critical systems [7] have shown that sign changes of the force can be achieved by varying boundary fields [8,9]. Sign control has been achieved experimentally with judicious choice of materials in case of QED Casimir forces [10], and with appropriate boundary conditions for critical FIF [11–13].

The non-additive character of FIF has also prompted a quest for reversing the sign of Casimir forces solely by manipulation of shapes. The original impetus comes from the intriguing result by Boyer [14] for the modification of QED zero point energy by a spherical metal shell. The suggestion that this result may imply repulsion between two hemispheres was later ruled out by a general theorem for attraction between mirror symmetric shapes [15,16]. There are indeed specific geometrical arrangements in which the normally attractive QED force in vacuum appears repulsive when constrained along a specific axis (e.g. [17,18]), but is unstable when moved off such axis. Indeed, a generalized Earnshaw's theorem for FIF in QED rules out the possibility of stable levitation (and consequently force reversals) in most cases [19].

Two dimensional (2D) membranes have provided yet another arena for investigation of FIF, mostly focused on interactions arising due to modifications of capillary fluctuations (see, e.g. [20,21] and references therein). More recently, motivated by the possibility that the lipid mixtures composing biological membranes are poised at criticality [22,23], it has been proposed that inclusions (such as proteins) on such membranes are subject to 2D analogs of critical FIF [24]. A notable advantage is that 2D systems at criticality can be described by conformal field theories (CFT) [25,26]: Casimir forces in a strip are related to the central charge of the CFT [27–29], with appropriate modification for boundaries. There are results for interactions between circles [24], needles [30]; Ref. [31] describes any compact shapes. Here, we consider the interaction between two wedges, or an array of wedges, as depicted in Fig. 1. We show that (with appropriate choice of CFT and boundary conditions) the FIF can be attractive or repulsive depending on the angle of the wedge; and that stable equilibrium can be obtained with truncated wedges and arrays of them.

Consider two identically corrugated, infinite boundaries that enclose a critical classical medium (e.g., a fluid or magnetic system at its critical temperature T_c) described by a CFT. The boundaries, S_1 and S_2, impose conformally invariant boundary conditions a and b, respectively, on the medium. While our method is applicable to any shape, as specific examples we study the periodic, wedge-like shapes in Figs. 1(b), (d). As interactions at proximity

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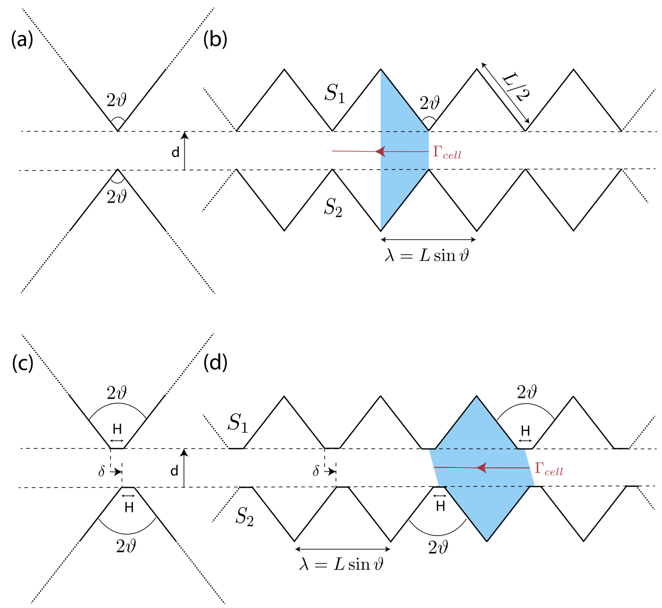

Fig. 1. Shapes considered: (a) two wedges, (b) strip with triangular corrugations, (c) truncated wedges with lateral shift, (d) strip with truncated corrugations and lateral shift. The blue regions mark half a unit cell (b) and a full unit cell (d). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

are dominated by the tips, we also consider the infinite wedges depicted in Figs. 1(a), (c). Following our approach for compact shapes [31], the strip with deformed boundaries is conformally mapped to a flat strip. Information about the intervening medium enters only via its conformal charge c, and the scaling dimension h_{ab} of the boundary changing operator (BCO) from a to b; with $h_{ab} = 0$ for like boundaries [32]. All information about the shape of the deformed strip is encoded in the conformal map to the flat strip. This map, and hence the FIF, can be obtained from the solution to an electrostatic problem. In the following, we combine the normal (y) and lateral (x) components of the force into the complex expression $F = (F_x - iF_y)/2$. For periodically deformed boundaries with wavelength λ and length $W \to \infty$, $F = F_{\rm strip} + F_{\rm geo}$, where the first contribution is the force on a strip,[1]

$$F_{\rm strip} = -i\,\frac{\pi}{2}\left(\frac{c}{12} - \tilde\eta\right)\frac{W}{\lambda}\frac{1}{2\ell^2}\int_{\Gamma_{\rm cell}} (\partial_z w)^2 dz, \tag{1}$$

that is determined by the free energy (per unit length) $\mathcal{F}_{\rm strip} = -(\pi/2)(c/12 - \tilde\eta)/\ell$ of a flat strip of width $\ell = 2\pi/C_{\rm cell}$ with $C_{\rm cell}$ the electrostatic capacitance of the deformed strip per unit cell, and $\tilde\eta \equiv 2h_{ab}$. The second contribution is the geometric force

$$F_{\rm geo} = -\frac{ic}{24\pi}\frac{W}{\lambda}\int_{\Gamma_{\rm cell}} \{w, z\} dz, \tag{2}$$

where $\{w, z\} \equiv (\partial_z^3 w/\partial_z w) - (3/2)(\partial_z^2 w/\partial_z w)^2$ is the Schwarzian derivative of the conformal map $w(z)$ of the deformed to the flat strip [32]. Due to periodicity, it is sufficient to construct $w(z)$ for a unit cell so that integrations in Eqs. (1), (2) are restricted to a path $\Gamma_{\rm cell}$ that separates S_1 and S_2 within a unit cell [cf. Fig. 1]. Of course, the forces are proportional to the number of unit cells, W/λ.[2] Whereas the strip force depends on shape simply via the electrostatic capacitance [31], the geometric force has a more intricate dependence on shape.

Conformal maps are physically realized as equipotential curves and stream lines in electrostatics. We employ this analogy to derive a general result for the Casimir force in terms of the electrostatic potential $U(x, y)$ on the strip with the two boundaries held at a fixed potential difference $\Delta U = 1$. The conformal map is then given by $w(z) = U + iV$ where V is the conjugate harmonic function to U. Clearly $\ell = \Delta U = 1$. Since Eqs. (1), (2) involve only derivatives of $w(z)$, we use the Cauchy–Riemann equations to get $\partial_z w = \partial_x U - i\partial_y U$ and eliminate V. For practical computations (e.g. using finite element solvers) it is useful to express the Casimir force in terms of line integrals of real valued vector fields that are fully determined by derivatives of U. Parameterizing the contour $\Gamma_{\rm cell}$ by $\mathbf{r}(s) = [x(s), y(s)]$ for $0 \le s \le 1$, and splitting into real and imaginary parts, we obtain the force in terms of c, $\tilde\eta$ and U as

$$F_{\rm strip} = \frac{\pi}{2}\left(\frac{c}{12} - \tilde\eta\right)\frac{W}{\lambda}\left\{\int_0^1 \mathbf{F}_1[\mathbf{r}(s)]\cdot\mathbf{r}'(s)ds + i\int_0^1 \mathbf{F}_2[\mathbf{r}(s)]\cdot\mathbf{r}'(s)ds\right\} \tag{3}$$

$$F_{\rm geo} = -\frac{ic}{24\pi}\frac{W}{\lambda}\left\{\int_0^1 \mathbf{G}_1[\mathbf{r}(s)]\cdot\mathbf{r}'(s)ds + i\int_0^1 \mathbf{G}_2[\mathbf{r}(s)]\cdot\mathbf{r}'(s)ds\right\}$$

with the vector fields (4)

$$\mathbf{F}_1 = \begin{pmatrix} -\partial_x U\partial_y U \\ \frac{1}{2}\left((\partial_x U)^2 - (\partial_y U)^2\right)\end{pmatrix},\quad \mathbf{F}_2 = \begin{pmatrix} -\frac{1}{2}\left((\partial_x U)^2 - (\partial_y U)^2\right) \\ -\partial_x U\partial_y U\end{pmatrix} \tag{5}$$

$$\mathbf{G}_1 = \frac{1}{\left((\partial_x U)^2 + (\partial_y U)^2\right)^2} \times \begin{pmatrix} \frac{1}{2}\left(\partial_x^2 U\partial_y U - \partial_x\partial_y U\partial_x U\right)^2 - \frac{1}{2}\left(\partial_x^2 U\partial_x U + \partial_x\partial_y U\partial_y U\right)^2 \\ \left(\partial_x^2 U\partial_x U + \partial_x\partial_y U\partial_y U\right)\left(\partial_x^2 U\partial_y U - \partial_x\partial_y U\partial_x U\right)\end{pmatrix} \tag{6}$$

$$\mathbf{G}_2 = \frac{1}{\left((\partial_x U)^2 + (\partial_y U)^2\right)^2} \times \begin{pmatrix} -\left(\partial_x^2 U\partial_x U + \partial_x\partial_y U\partial_y U\right)\left(\partial_x^2 U\partial_y U - \partial_x\partial_y U\partial_x U\right) \\ \frac{1}{2}\left(\partial_x^2 U\partial_y U - \partial_x\partial_y U\partial_x U\right)^2 - \frac{1}{2}\left(\partial_x^2 U\partial_x U + \partial_x\partial_y U\partial_y U\right)^2\end{pmatrix}. \tag{7}$$

We note that the strip force $F_{\rm strip}$ is proportional to the usual electrostatic force. This result also implies that the critical Casimir force between any pair of mirror symmetric boundaries is attractive for $c > 0$ [15,16]: In this case the electrostatic potential must be constant along the x-axis of mirror symmetry. Choosing this axis as $\Gamma_{\rm cell}$ gives $\mathbf{r}'(s) \sim -\hat{\mathbf{x}}$ and hence shows that both $F_{\rm strip}$ and $F_{\rm geo}$ have a vanishing real part and a negative imaginary part for $c/12 - \tilde\eta > 0$, which includes like boundaries ($\tilde\eta = 0$). This implies a

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[1] Throughout the paper we measure energies in units of $k_B T_c$; correspondingly forces are proportional to $k_B T_c$ divided by an appropriate length scale.

[2] For finite W the corrections to the force are exponentially small in W with the characteristic scale λ. This can be seen by mapping the finite strip to a cylinder [31].