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Global surpluses of spin-base invariant fermions

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ABSTRACT

The spin-base invariant formalism of Dirac fermions in curved space maintains the essential symmetries of general covariance as well as similarity transformations of the Clifford algebra. We emphasize the advantages of the spin-base invariant formalism both from a conceptual as well as from a practical viewpoint. This suggests that local spin-base invariance should be added to the list of (effective) properties of (quantum) gravity theories. We find support for this viewpoint by the explicit construction of a global realization of the Clifford algebra on a 2-sphere which is impossible in the spin-base non-invariant vielbein formalism.

1. Introduction

The mutual interrelation of matter and spacetime ("matter curves spacetime – spacetime determines the paths of matter") is particularly apparent for fermions. For instance for Dirac fermions, information about both spin as well as spacetime meets in the Clifford algebra,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}, \tag{1}$$

where the Dirac matrices γ_μ as well as the metric $g_{\mu\nu}$ generally are spacetime dependent. While many tests of classical gravity rely on vacuum solutions to Einstein's equation, also many attempts at quantizing gravity primarily concentrate on the dynamics of spacetime without matter, cf. [1]. This is similar in spirit to "quenched" QCD which allows to understand already many features of the strong interactions at the quantum level even quantitatively. Only recently, some evidence has been collected that the existence of matter degrees of freedom can constrain the existence of certain quantum gravity theories [2–6]. This is again analogous to QCD where the presence of too many dynamical fermions can destroy the high-energy completeness of the theory.

The interrelation of gravity and fermions provided by (1) has also been interpreted in various partly conflicting directions: read from right to left, one is tempted to conclude that one first needs a spacetime metric $g_{\mu\nu}$ in order to give a meaning to spinorial degrees of freedom and corresponding physical observables such as currents $\sim \bar{\psi}\gamma_\mu\psi$. On the other hand, representation theory of the Lorentz group in flat space suggests that all nontrivial representations can be composed out of the fundamental spinorial representation, culminating into (1) for Dirac spinors. If so, then also the metric might be a composite degree of freedom, potentially arising as an expectation value of composite spinorial operators, see, e.g., [7–9].

As a starting point to disentangle this hen-or-egg problem – spinors or metric first? – we consider the Clifford algebra (1) as fundamental in this work. We emphasize that this is different from a conventional approach [10], where one starts from the analogous Clifford algebra in flat (tangential) space, $\{\gamma_{(\mathrm{f})_a}, \gamma_{(\mathrm{f})_b}\} = 2\eta_{ab}\mathrm{I}$, with fixed $\gamma_{(\mathrm{f})_a}$ and then uplifts the Clifford algebra to curved space with the aid of a *vielbein* $e_\mu{}^a(x)$, such that $\gamma_{(\mathrm{e})_\mu} = e_\mu{}^a\gamma_{(\mathrm{f})_a}$ satisfies (1). In addition to diffeomorphism invariance, the vielbein approach supports a local SO(3,1) symmetry of Lorentz transformations in tangential space, i.e. with respect to the roman *bein* index. By contrast, the Clifford algebra (1) actually supports a bigger symmetry of local similarity (spin-base) transformations in addition to general covariance.

Developing a formalism that features this full spin-base invariance has first been initiated by Schrödinger [11] and amended with the required spin metric by Bargmann [12] in 1932. Surprisingly, it has been rarely used in the literature, see, e.g., [13–19], or even reinvented [20]. A full account of the formalism also including spin torsion has recently been given in [21]. Particular advantages are not only the inclusion and generalization of the vielbein formalism. In a quantized setting, it even justifies the widespread use of the vielbein as an auxiliary quantity and not as a fundamental entity. Common quantization schemes relying on the metric as fundamental degree of freedom remain applicable also with fermionic matter. Hence, a Jacobian from the variable transformation to the vielbein does not have to be accounted for [21].

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In this work, we present further advantages of the spin-base invariant formalism and discuss some general aspects in order to elucidate the interplay between diffeomorphisms and spin-base transformations. We point out various options of defining the spin-base group, differing by the possible field content of further interactions and also naturally permitting a Spin^c structure. Since the conventional vielbein formalism can always be recovered within the spin-base invariant formalism, it is tempting to think that the latter is merely a technical perhaps overabundant generalization of the former. We demonstrate that this is not the case by an explicit construction of a global spin-base on the 2-sphere – a structure which is not possible in the conventional formalism because of global obstructions from the Poincaré–Brouwer (hairy-ball) theorem. We believe that this example is paradigmatic for the surpluses of the spin-base invariant formalism.

2. General covariance and spin-base invariance

Local symmetries are expected to be fundamental, since symmetry-breaking perturbations typically contain relevant components which inhibit symmetry emergence. Hence, we consider the local symmetries of the Clifford algebra as fundamental. These are diffeomorphisms (formalized by tensor calculus of the Greek indices) and local similarity transformations of the Dirac matrices [22], the spin base transformations,

$$\gamma_\mu \to \mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \quad \psi \to \mathcal{S}\psi, \quad \bar\psi \to \bar\psi\mathcal{S}^{-1}. \tag{2}$$

The γ_μ transformation leaves the Clifford algebra equation (1) invariant. The corresponding transformation of spinors ensures that typical fermion bilinears and higher-order interaction terms serving as building blocks for a relativistic field-theory are also invariant, provided a suitable connection exists. The latter should obey

$$\Gamma_\mu \to \mathcal{S}\Gamma_\mu\mathcal{S}^{-1} - (\partial_\mu\mathcal{S})\mathcal{S}^{-1}, \tag{3}$$

such that $\nabla_\mu = \partial_\mu + \Gamma_\mu$ forms a covariant derivative with the standard covariance properties with respect to both diffeomorphisms as well as spin base transformations. The connection Γ_μ has explicitly been constructed in $d = 4$ dimensions [20,21] as well as in lower [23] and higher dimensions [24]. For vanishing spin torsion [21], the traceless part of Γ_μ can fully be expressed in terms of the Dirac matrices and their first derivatives (part of the terms can be summarized by Christoffel symbols).

For simplicity, let us confine ourselves to the cases $d = 4$ and $d = 2$ (for generalizations, see [24]). Here, the dimension of the irreducible representation of the Clifford algebra is $d_\gamma = 4$ and $d_\gamma = 2$, respectively. A natural choice for the group of spin base transformations maintaining all invariance properties mentioned above is then given by $\mathrm{GL}(d_\gamma, \mathbb{C})$.

However, $\mathrm{GL}(d_\gamma, \mathbb{C})$ contains continuous subgroups that act trivially on the Clifford algebra. Considering the invariance properties of the Clifford algebra as fundamental, trivial subgroups appear redundant. Locally, elements of $\mathrm{GL}(d_\gamma, \mathbb{C})$ can be decomposed into an $\mathrm{SL}(d_\gamma, \mathbb{C})$ element and two factors proportional to the identity: a phase $\in \mathrm{U}(1)$ and a modulus $\in \mathbb{R}_+$. Confining ourselves to the nontrivial invariance properties, hence suggest to identify the set of transformation matrices $\mathcal{S}$ with the fundamental representation of $\mathrm{SL}(d_\gamma, \mathbb{C})$. This special linear group still has redundancies as its discrete center $\mathbb{Z}_{d_\gamma}$ does not transform the Dirac matrices nontrivially.

The choice of the local spin-base group becomes only relevant, once a dynamics is associated with the connection. For the choice of $\mathrm{SL}(d_\gamma, \mathbb{C})$ and vanishing torsion, the corresponding field strength $\Phi_{\mu\nu}$ satisfies the identity [20,21]

$$\Phi_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \frac{1}{8} R_{\mu\nu\lambda\kappa}[\gamma^\lambda, \gamma^\kappa]. \tag{4}$$

It is somewhat surprising as well as reassuring that – out of the large number of degrees of freedom in Γ_μ – only those acquire a nontrivial dynamics which can be summarized in the Christoffel symbols and hence lead to the Riemann tensor on the right-hand side of Eq. (4). As a consequence, spin-base invariance is also a (hidden) local symmetry of any special relativistic fermionic theory in flat space with an automatically trivial dynamics for the connection, even if kinetic terms of the form $\sim \mathrm{tr}\, \gamma^\mu \Phi_{\mu\nu}\gamma^\nu$ ($\sim R$ Einstein–Hilbert) or $\sim \mathrm{tr}\, \Phi_{\mu\nu}\Phi^{\mu\nu}$ would be added.

This is different if spin-base transformations are associated with $\mathrm{GL}(d_\gamma, \mathbb{C})$. Then two additional abelian field strengths corresponding to the U(1) and the non-compact $\mathbb{R}_+$ factors appear on the right-hand side of Eq. (4) and thus introduce further physical degrees of freedom. These correspond to the imaginary and real part of the trace of the connection Γ_μ.[1] Hence, the identification of the spin base group is in principle an experimental question to be addressed by verifying the interactions of fermions. In this sense, one might speculate whether the hypercharge U(1) factor of the standard model could be identified with the spin-base group provided proper charge assignments are chosen for the different fermions. The inclusion of the U(1) factor is particularly natural on manifolds that do not permit a Spin structure (e.g., CP^2) [25], as it provides exactly for the necessary ingredient to define the more general Spin^c structure.

For the remainder of this work, it suffices to consider $\mathrm{SL}(d_\gamma, \mathbb{C})$ as the group of spin-base transformations. Returning to the hen-or-egg problem, Eq. (4) seems interpretable as another manifestation of the intertwining of Dirac structure and curvature, or spin-base and general covariance. However, a clearer picture arises from an explicit coordinate transformation of the Clifford algebra,

$$\{\gamma'_\mu, \gamma'_\nu\} = 2g'_{\mu\nu}\mathrm{I} = 2\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda}\mathrm{I} = \left\{\frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \frac{\partial x^\lambda}{\partial x'^\nu}\gamma_\lambda\right\}. \tag{5}$$

Read together with the spin-base invariance of the Clifford algebra [22,26], Eq. (5) implies that the most general coordinate transformation of a Dirac matrix is given by

$$\gamma_\mu \to \gamma'_\mu = \frac{\partial x^\rho}{\partial x'^\mu}\mathcal{S}\gamma_\rho\mathcal{S}^{-1}. \tag{6}$$

From the sheer size of the spin-base group (at least $\mathrm{SL}(d_\gamma, \mathbb{C})$), it is obvious that this is a larger set of Dirac matrices satisfying the Clifford algebra than can be spanned by the vielbein construction. In the latter, only those realizations of the Clifford algebra $\gamma_{(e)\mu}$ are considered, that can be spanned by a fixed set of Dirac matrices, $\gamma_{(e)\mu} = e_\mu{}^a\gamma_{(\mathrm{f})a}$. A local Lorentz transformation with respect to the *bein* index can then be rewritten in terms of

$$\Lambda_a{}^b\gamma_{(\mathrm{f})b} = \mathcal{S}_{\mathrm{Lor}}\gamma_{(\mathrm{f})a}\mathcal{S}_{\mathrm{Lor}}^{-1}, \tag{7}$$

where $\mathcal{S}_{\mathrm{Lor}} \in \mathrm{Spin}(d-1,1) \subset \mathrm{SL}(d_\gamma, \mathbb{C})$. Conventionally, the $\mathcal{S}_{\mathrm{Lor}}$ factors are interpreted as Lorentz transformations of Dirac spinors, e.g., $\psi \to \mathcal{S}_{\mathrm{Lor}}\psi$. This way of interpreting the Lorentz subgroup of spin-base transformations is at the heart of understanding fields as representations of the Lorentz group. This viewpoint is held to argue that higher-spin fields (such as the metric) may eventually be composed out of a fundamental spinorial representation.

However, there is no such simple relation as Eq. (7) for general coordinate transformations. This is already obvious in flat space:

[1] The non-compact factor (real part of $\mathrm{tr}\,\Gamma_\mu$) can be removed by fixing the determinant of the spin metric, see [20,21].