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A divergence-free method to extract observables from correlation functions

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ABSTRACT

Correlation functions provide information on the properties of mesons in vacuum and of hot nuclear matter. In this work, we present a new method to derive a well-defined spectral representation for correlation functions. Combining this method with the quark gap equation and the inhomogeneous Bethe–Salpeter equation in the rainbow-ladder approximation, we calculate in-vacuum masses of light mesons and the electrical conductivity of the quark–gluon plasma. The analysis can be extended to other observables of strong-interaction systems.

1. Introduction

Hadrons contribute to most of the visible matter in our real world and are undoubtedly an embodiment of dynamical chiral symmetry breaking (DCSB) and confinement. Current and future hadron physics facilities are focusing on hadron spectroscopy in order to shed light on the mysteries of quantum chromodynamics (QCD). On the other hand, it is believed that the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) are able to create the quark–gluon plasma (QGP) state of the early Universe through a "mini-big bang". This provides us with the possibility to study quark–gluon dynamics directly and to enrich our understanding of the QCD phase diagram. The transport coefficients of the QGP, which directly reflect details of the quark–gluon interaction, are highly interesting from both experimental and theoretical viewpoints.

A unified description for physics in the two areas has been a central goal and great challenge for decades. Lattice QCD which is based on Monte Carlo simulations of quantum fields on finite discrete spacetime lattices has achieved numerous significant results, however, it also has its own limitations [1,2]. Thus, relativistically covariant formalisms of continuum quantum field theory (QFT) are still desirable. Among them, the Dyson–Schwinger equation (DSE) approach [3–5] is a framework that includes both DCSB and confinement [6]. Remarkably, at zero temperature, $T = 0$, a single DSE interaction kernel preserving the one-loop renormalization group behavior of QCD has been able to provide a unified description of the pion's electromagnetic form factor [7], its valence-quark distribution amplitude [8], and numerous other quantities [9,10]. Therefore, it is of great significance to extend the DSE approach to further quantitative studies of hadron and QGP physics.

In the DSE framework, hadrons, i.e., color-singlet bound states of quarks, are described by the Bethe–Salpeter equation (BSE) or the Faddeev equation. Solving these equations requires the quark propagator, i.e., the solution of the gap equation, on the complex momentum plane. The analytical structure of the quark propagator strongly depends on the specified truncation scheme and interaction model. This may lead to technical difficulties in the study of light-quark hadrons with masses above 1 GeV and meson bound-states composed of one heavy and one light valence-quark. Those aspects of these problems connected with continuation into the complex plane can be solved using the perturbation theory integral technique [11,12], as illustrated in Ref. [7], whereas, as highlighted elsewhere [13], resolving the difficulties associated with heavy-light mesons requires bound-state kernels which are more sophisticated than that obtained in the simplest DSE truncation. At nonzero temperature, $T \neq 0$, Matsubara frequencies are introduced

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in imaginary-time thermal field theory [14]. Then, the situation is even more complicated since we do not know how to analytically continue Matsubara frequencies. Thus, it is a long-standing challenge to study in-medium hadrons.

At $T \neq 0$, transport coefficients can be calculated from meson spectral functions through Kubo formulae. Solving for meson spectral functions, one has to calculate Euclidean meson correlation functions. However, in terms of Green functions, the calculations are highly divergent. As we will see, the subtraction scheme which works at $T = 0$ is not applicable at $T \neq 0$. Thus, the divergence problem precludes the study of transport properties.

In this paper, we propose a novel approach based on spectral analysis, which can systematically solve the problems mentioned before. Using our new approach, we can extend the DSE study to a much wider range of applications. To demonstrate this, we calculate the masses of the π- and ρ-meson in vacuum and the electrical conductivity of the QGP with a single DSE interaction kernel. Both the result for the electrical conductivity and the approach itself are essentially new.

2. Meson correlation functions

The retarded correlation function of local meson operators is defined as

$$\Pi_H^R(t,\vec{x}) = \langle J_H(t,\vec{x}) J_H^\dagger(0,\vec{0})\rangle_\beta, \tag{1}$$

where $\beta = 1/T$ and $\langle ...\rangle_\beta$ denotes the thermal average. The operator J_H has the following form

$$J_H(t,\vec{x}) = \bar{q}(t,\vec{x})\gamma_H q(t,\vec{x}), \tag{2}$$

with $\gamma_H = \mathbf{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu$ for scalar, pseudo-scalar, vector, and axial-vector channels, respectively. The meson spectral function is related to the imaginary part of the Fourier transform of the retarded meson correlation function [15], namely,

$$\rho_H(\omega,\vec{p}) = 2\,\mathrm{Im}\,\Pi_H^R(\omega,\vec{p}). \tag{3}$$

Note that the spectral function is positive semi-definite for positive frequency and that $\rho_H(\omega,\vec{0}) = -\rho_H(-\omega,\vec{0})$. In the zero-momentum limit, $\vec{p} = \vec{0}$, the Euclidean correlation function which can be connected with the retarded correlation function by analytic continuation, i.e., $\omega + i\epsilon \to i\omega_n$, has the following spectral representation,

$$\Pi_H(\omega_n^2) = \int_0^\infty \frac{d\omega^2}{2\pi}\frac{\rho_H(\omega)}{\omega^2+\omega_n^2} - (\text{subtraction}), \tag{4}$$

where $\omega_n = 2n\pi T$, $n \in Z$, are the bosonic Matsubara frequencies. Note that an appropriate subtraction is required because the spectral integral in Eq. (4) does not converge, i.e., $\rho_H(\omega\to\infty) \propto \omega^2$ (see, e.g., Eq. (10) below).

Using the Fourier transform on Eq. (4), one can obtain the spectral representation of the Euclidean temporal correlation functions without any subtraction. Lattice QCD generally adopts such a form [16]. However, it is not applicable for the DSE approach. As we will see, the numerical calculation of the Fourier transform is actually very difficult because of divergences in computing $\Pi_H(\omega_n^2)$ by the DSE approach. At $T = 0$, one has the so-called twice-subtracted dispersion relation [17] which is well-defined. At $T \neq 0$, its straightforward extension reads

$$\Pi_H(\omega_n^2) = \Pi_H(0) + \omega_n^2\Pi_H'(0) + \int_0^\infty \frac{d\omega^2}{2\pi}\frac{\omega_n^4\rho_H(\omega)}{\omega^4(\omega^2+\omega_n^2)}. \tag{5}$$

The above equation takes care of the ultraviolet divergence. However, it generates a divergence in the infrared region because $\rho_H(\omega\to 0) \propto \omega$ at $T \neq 0$. Moreover, Eq. (5) is correct only if the derivatives of the Euclidean and retarded correlators can be connected by analytical continuation. It can be proved that such an analytical continuation does not hold at $T \neq 0$. At one-loop level, one can easily check that the analytical continuation breaks down for the zeroth component of the vector correlation function. Thus, Eq. (5) is ill-defined and useless at $T \neq 0$.

Here we would like to present a new method to construct a well-defined spectral representation. We introduce a transform for a function $f(x)$,

$$\hat{\mathcal{O}}_N(x_1,\ldots,x_N)\{f\} = \sum_{i=1}^N f(x_i)\prod_{j\neq i}^N \frac{1}{x_i - x_j}, \tag{6}$$

where $x_1 \neq x_2 \neq \ldots \neq x_N$. If $f(x)$ is an N-order polynomial, then $\hat{\mathcal{O}}_{N+2}\{f\} = 0$, e.g., $\hat{\mathcal{O}}_3\{\text{linear function}\} = 0$. According to analytical properties of correlation functions in QFT [18], the subtractions in the dispersion relations are always polynomials of momenta (or Matsubara frequencies), e.g., the subtraction for the meson correlation function is a linear function of ω_n^2. Thus, using the 3-order transform for $\Pi_H(\omega_n^2)$ in Eq. (4) or (5), i.e.,

$$\begin{aligned}\hat{\Pi}_H(\omega_i^2,\omega_j^2,\omega_k^2) &= \hat{\mathcal{O}}_3(\omega_i^2,\omega_j^2,\omega_k^2)\{\Pi_H\} \\ &= \frac{\Pi_H(\omega_i^2)}{(\omega_i^2-\omega_j^2)(\omega_i^2-\omega_k^2)} \\ &\quad + \frac{\Pi_H(\omega_j^2)}{(\omega_j^2-\omega_i^2)(\omega_j^2-\omega_k^2)} \\ &\quad + \frac{\Pi_H(\omega_k^2)}{(\omega_k^2-\omega_i^2)(\omega_k^2-\omega_j^2)},\end{aligned} \tag{7}$$

where $\omega_{i,j,k}$ are arbitrary unequal Matsubara frequencies, one finds that the subtraction in Eq. (4) or the linear term of ω_n^2 in Eq. (5) is canceled. Correspondingly, $\hat{\Pi}_H(\omega_i^2,\omega_j^2,\omega_k^2)$ can be expressed as the surviving integral of the spectral function,

$$\hat{\Pi}_H(\omega_i^2,\omega_j^2,\omega_k^2) = \int_0^\infty \frac{d\omega^2}{2\pi}\frac{\rho_H(\omega)}{(\omega^2+\omega_i^2)(\omega^2+\omega_j^2)(\omega^2+\omega_k^2)}. \tag{8}$$

Note that Eq. (8) is a novel version of the spectral representation for meson correlation functions. As we mentioned before, it is found that the traditional spectral representations, i.e., Eqs. (4) and (5), are not well-defined because of the infrared or ultraviolet divergence. However, through a simply power analysis, one can easily verify that the integral in Eq. (8) is divergence-free both in the ultraviolet and infrared regions. Furthermore, Eq. (7) is an exact algebraic equation without any approximation. Therefore, by analytic continuation, Eq. (8) is actually consistent with the original definition of the spectral function, i.e., Eq. (3). As we will see, since Eq. (8) is a well-defined expression directly formulated in frequency (or momentum) space, it is very suitable for analyzing Euclidean Green functions obtained by nonperturbative functional frameworks, e.g., the DSE approach. Namely, using Eq. (8) one is able to extract observables which are encoded in the spectral functions from the nonperturbatively calculated correlation functions.

At first glance, Eq. (8) depends on three different frequencies (or momenta) and thus is a complicated three-dimensional equation. But it can be simplified easily. For numerical convenience, one can further introduce a one-variable correlator as $\tilde{\Pi}_H(\omega_i^2) = \hat{\Pi}_H(\omega_i^2,\omega_{i+1}^2,\omega_{i+2}^2)$ (where $\omega_{i+1} = \omega_i + 2\pi T$ and $\omega_{i+2} = \omega_i +$