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Gauge independence in a higher-order Lagrangian formalism via change of variables in the path integral

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ABSTRACT

In this paper we work out the explicit form of the change of variables that reproduces an arbitrary change of gauge in a higher-order Lagrangian formalism.

1. Introduction

It is a standard lore in the path integral formalism, that any result (such as, *e.g.*, the Schwinger–Dyson equations, the Ward identities, etc.), that can be (formally) proven via change of integration variables, can equivalently be (formally) obtained via an integration by parts argument. And vice-versa. The latter method is typically the simplest. In 1996 it was shown in Ref. [1], by using integration by parts, how to formulate a higher-order field–antifield formalism that is independent of gauge choice. In this paper we work out the explicit form of the change of variables that reproduces a given change of gauge in a higher-order formalism. Perhaps not surprisingly, the construction relies on identifying appropriate homotopy operators.

2. The Δ operator

From a modern perspective [2] the primary object in the Lagrangian field–antifield formalism [3–5] is the Δ **operator**, which is a nilpotent Grassmann-odd differential operator

$$\Delta^2 = 0, \qquad \varepsilon(\Delta) = 1, \tag{2.1}$$

and which depends on antisymplectic variables z^A and their corresponding partial derivatives ∂_B. Their commutator[1] reads

$$\left[\overrightarrow{\partial}_B, z^A\right] = \delta^A_B. \tag{2.2}$$

3. *Sp*(2)-symmetric formulation

We mention for completeness that there also exists an $Sp(2)$-symmetric Lagrangian field–antifield formulation [6]. This formulation is endowed with two Grassmann-odd nilpotent, anticommuting Δ^a operators

$$\Delta^{\{a}\Delta^{b\}} = 0, \qquad \varepsilon(\Delta^a) = 1,$$
$$a, b \in \{1, 2\}. \tag{3.1}$$

Often (but not always!) the resulting $Sp(2)$-symmetric formulas look like the standard formulas with $Sp(2)$-indices added and symmetrized in a straightforward manner. In this paper, we will usually focus on the standard formulation and only mention the corresponding $Sp(2)$-symmetric formulation when it deviates in a non-trivial manner.

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[1] The word *super* is often implicitly implied. For instance, the word *commutator* means the supercommutator $[F, G] \equiv FG - (-1)^{\varepsilon_F \varepsilon_G} GF$.

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4. Planck number grading for operators

Planck's constant $\hbar$ is here treated as a formal parameter (as opposed to an actual number) in the spirit of deformation quantization (as opposed to geometric quantization). The *Planck number grading* Pl for operators is defined via the rules

$$\mathrm{Pl}(\hbar)=1, \qquad \mathrm{Pl}(z^A)=0, \qquad \mathrm{Pl}(\overrightarrow{\partial}_A)=-1, \tag{4.1}$$

and extended to normal-ordered[2] differential operators in the natural way. More precisely, a derivative ∂_A inside an operator F gets assigned Planck number -1 (0) for the parts that act outside (inside) the operator, respectively. We mention for later convenience the superadditivity of Planck number grading

$$\mathrm{Pl}(FG)\geq \mathrm{Pl}(F)+\mathrm{Pl}(G),$$
$$\mathrm{Pl}([F,G])\geq \mathrm{Pl}(F)+\mathrm{Pl}(G)+1, \tag{4.2}$$

where the uppercase letters F and G denote operators.

5. Higher-order Δ operator

In the standard field–antifield formalism [3–5], the Δ operator is a second-order operator. (See also Section 18.) In the higher-order generalization [1], which is the main topic of this paper, the Δ operator is assumed to have Planck number grading [7]

$$\mathrm{Pl}(\Delta)\geq -2. \tag{5.1}$$

Evidently, the Planck number inequality (5.1) means that the normal-ordered Δ operator is of the following triangular form[3]

$$\Delta=\sum_{n=-2}^{\infty}\sum_{m=0}^{n+2}\left(\frac{\hbar}{i}\right)^n\Delta_{n,m},$$
$$\Delta_{n,m}=\Delta_{n,m}^{A_1\ldots A_m}(z)\overrightarrow{\partial}_{A_m}\ldots\overrightarrow{\partial}_{A_1}. \tag{5.2}$$

The higher-order terms in the Δ operator can, *e.g.*, be physically motivated as quantum corrections, which arise in the correspondence between the path integral and the operator formalism.

6. Path integral

The (formal) path integral

$$Z_X=\int d\mu\, w x, \quad w\equiv e^{\frac{i}{\hbar}W},\ x\equiv e^{\frac{i}{\hbar}X}, \tag{6.1}$$

in the W–X-formalism [8–14] consists of three parts:

1. A *path integral measure* $d\mu=\rho[dz][d\lambda]$, where λ^α are Lagrange multipliers implementing the gauge fixing conditions, and $z^A\equiv\{\phi^\alpha;\phi^*_\alpha\}$ are the antisymplectic variables, *i.e.*, fields ϕ^α and antifields ϕ^*_α. Here $\rho=\rho(z)$ is a density with $\varepsilon(\rho)=0$ and $\mathrm{Pl}(\ln\rho)\geq -1$.
2. A gauge-generating *quantum master action* W, which satisfies the *quantum master equation* (QME)[4]

$$(\Delta w)=0, \qquad w\equiv e^{\frac{i}{\hbar}W}, \qquad \mathrm{Pl}(W)\geq 0. \tag{6.2}$$

The path integral (6.1) will in general depend on W, since W contains all the physical information about the theory, such as, *e.g.*, the original action, the gauge generators, etc. [15,16]. The triangular form (5.2) of the Δ operator implies that the QME (6.2) is perturbative in Planck's constant $\hbar$, *i.e.*,

$$\mathrm{Pl}\left(w^{-1}\Delta(\hbar,z,\partial)w\right)=\mathrm{Pl}\left(\Delta\left(\hbar,z,\partial+\frac{i}{\hbar}(\partial W)\right)\right)\geq -2. \tag{6.3}$$

Besides the triangular form (5.1), which is imposed to ensure perturbativity, there are additional "boundary" and rank conditions to guarantee the pertinent classical[5] master equation and proper classical master action S [15,16].

3. A gauge-fixing *quantum master action* X, which satisfies the transposed quantum master equation

$$(\Delta^T x)=0, \qquad x\equiv e^{\frac{i}{\hbar}X}, \qquad \mathrm{Pl}(X)\geq 0. \tag{6.4}$$

The path integral (6.1) will in general not depend on X, cf. Section 13 and Section 16.

The *transposed operator* F^T has the property that

$$\int d\mu (F^T f)g=(-1)^{\varepsilon_f\varepsilon_F}\int d\mu\, f(Fg). \tag{6.5}$$

Here the lowercase letters $f,g,\ldots$ denote functions, while the upper case letters $F,G,\ldots$ denote operators. One can construct any transposed operator by successively apply the following rules

$$(F+G)^T=F^T+G^T, \qquad (FG)^T=(-1)^{\varepsilon_F\varepsilon_G}G^TF^T,$$
$$(z^A)^T=z^A, \qquad \partial_A^T=-\rho^{-1}\partial_A\rho. \tag{6.6}$$

In particular the transposed operator Δ^T is also nilpotent

$$(\Delta^T)^2=0. \tag{6.7}$$

The transposed derivative ∂_A^T satisfies a modified Leibniz rule:

$$\partial_A^T(fg)=(\partial_A^T f)g-(-1)^{\varepsilon_A\varepsilon_f}f(\partial_A g). \tag{6.8}$$

Let us mention for completeness that the Δ operator (which takes functions to functions) and the W–X-formalism can be recast in terms of Khudaverdian's operator Δ_E (which takes semidensities to semidensities) [17–25].

7. Higher-order quantum BRST operators

The quantum BRST operators σ_W and σ_X take operators into functions (*i.e.*, left multiplication operators). They are defined as

$$\sigma_W F:=\frac{\hbar}{i}w^{-1}([\Delta,F]w)\overset{(6.2)}{=}\frac{\hbar}{i}w^{-1}(\Delta F w), \tag{7.1}$$

$$\sigma_X F:=\frac{\hbar}{i}x^{-1}([\Delta^T,F]x)\overset{(6.4)}{=}\frac{\hbar}{i}x^{-1}(\Delta^T F x), \tag{7.2}$$

respectively, where F is an operator. They are nilpotent, Grassmann-odd,

$$\sigma_W^2=0=\sigma_X^2, \qquad \varepsilon(\sigma_W)=1=\varepsilon(\sigma_X), \tag{7.3}$$

and perturbative in the sense that

$$\mathrm{Pl}(\sigma_W F)\geq \mathrm{Pl}(F)\leq \mathrm{Pl}(\sigma_X F). \tag{7.4}$$

In the $Sp(2)$-symmetric formulation the quantum BRST operators σ_W^a and σ_X^a carry an $Sp(2)$-index since the Δ^a operator does.

[2] *Normal-ordering* means that all the z's appear to the left of all the ∂'s. *Antinormal-ordering* means the opposite.

[3] In contrast to the original proposal [1], we also allow the three terms $\Delta_{-2,0}$, $\Delta_{-1,0}$ and $\Delta_{-1,1}$ with negative n in Eq. (5.2). The two last terms arise naturally in the $Sp(2)$-symmetric formulation [6,12]. The two first terms affect the classical master equation. See also Sections 18–19 for the second-order case.

[4] The parenthesis in Eq. (6.2) is here meant to emphasize that the QME is an identity of functions (as opposed to differential operators), *i.e.*, the derivatives in Δ do *not* act outside the parenthesis. Note however that similar parenthesis will not always be written explicitly in order not to clog formulas. In other words, it must in general be inferred from the context whether an equality means an identity of functions or an identity of differential operators.

[5] The word *classical* means here independent of Planck's constant $\hbar$.