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BPS Wilson loop T-dual to spinning string in $AdS_5 \times S^5$

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ABSTRACT

We use the string sigma model action in $AdS_5 \times S^5$ to reconstruct the open string solution ending on the Wilson loop in $S^3 \times R$ parametrized by a geometric angle in S^3 and an angle in flavor space. Under the interchange of the world sheet space and time coordinates and the T-duality transformation with the radial inversion, the static open string configuration associated with the BPS Wilson loop with two equal angle parameters becomes a long open spinning string configuration which is produced by taking the special limit of equal two frequencies for the folded spinning closed string with two spins in $AdS_5 \times S^5$.

The AdS/CFT correspondence [1] has more and more revealed the deep relations between the $\mathcal{N}=4$ super Yang–Mills (SYM) theory and the string theory in $AdS_5 \times S^5$. A lot of fascinating results have been accumulated in the computation of the planar observables such as the spectrum, Wilson loops and scattering amplitudes [2].

By computing the expectation value of the Wilson loop consisting of a pair of antiparallel lines from the string theory in $AdS_5 \times S^5$ the effective potential between a pair of heavy W bosons has been extracted [3,4]. It has been investigated perturbatively in the gauge theory [5,6] and by the strong coupling expansion in the string theory [7–10] (see also [11]).

For the circular 1/2 BPS Wilson loop its expectation value evaluated in the string theory has been reproduced by performing the resummation of ladder diagrams in the $\mathcal{N}=4$ SYM theory [6,12] and using the localization arguments [13]. The lower supersymmetric Wilson loops on a two-sphere S^2 embedded into the R^4 spacetime have been analyzed by finding the corresponding open string solutions as well as by reducing a purely perturbative calculation in the soluble bosonic 2d Yang–Mills on the sphere [14–17].

In [18] the effective potential between a generalized quark antiquark pair has been computed by studying a family of Wilson loops in $S^3 \times R$ which are parametrized by two angle parameters ϕ and θ. The quark and antiquark lines are extended along the time direction and are separated by an angle $\pi - \phi$ on S^3. The parameter θ is the relative orientation of the extra coupling to the scalar field for the quark and the antiquark. Through the plane to cylinder transformation, the two lines in $S^3 \times R$ map into two half-lines with a cusp of angle $\pi - \phi$ in R^4, and the potential energy of static quark and antiquark is identical with the cusp anomalous dimension. The Wilson loop in $S^3 \times R$ with the Minkowski signature interpolates smoothly between the 1/2 BPS two antipodal lines at $\phi = 0$ and the coincident two antiparallel lines at $\phi = \pi$, while the two Wilson lines with a cusp in R^4 between the 1/2 BPS one straight line and the coincident two lines with a cusp of zero angle.

In the weak coupling expansion for the $\mathcal{N}=4$ SYM theory the effective potential has been computed at one-loop order [19] and at two-loop order for $\phi = 0$ [20], for $\phi \neq 0$ [18]. In the semiclassical expansion for the string theory by using the Nambu–Goto string action, the effective potential has been evaluated at leading order [19,14] and at one-loop order [18].

An exact formula for the Bremsstrahlung function of the cusp anomalous dimension for small values of ϕ and θ has been found by relating the cusp anomalous dimension to the localization result of certain 1/8 BPS circular Wilson loops [21] (see also [22]). The first two terms in the weak coupling and the strong coupling expansions of the exact formula agree with the results of the corresponding effective potential in [18]. The three-loop term in the weak coupling expansion has been produced by the explicit three-loop computation [23] and the three-loop expansion of the TBA equations [24,25]. There have been further studies about the cusp anomalous dimension associated with the cusped Wilson lines [26–28].

Suggested by a striking similarity between the Bremsstrahlung function [21] of the cusp anomalous dimension in the small angle limit and the slope function [29] found in the small spin limit

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of the AdS_5 folded string energy, there has been a construction of a possible relation between small (nearly point-like) closed strings in AdS_5 and long open strings ending at the boundary which correspond to nearly straight Wilson lines [30]. Through the T-duality along the boundary directions of Lorentzian AdS_5 in the Poincaré coordinates together with the radial inversion $z \to 1/z$ [32–34] and the interchange of space and time coordinates of the Minkowski world sheet, the world sheet of small closed string is related with the open string surface ending on wavy line representing small-velocity "quark" trajectory at the boundary. This open string solution corresponds to the small-wave open string solution in [35] which ends on a time-like near BPS Wilson loop differing by small fluctuations from a straight line. Further from the computation of the one-loop fluctuations about the classical small-wave open string solution, the one-loop correction to the energy radiated by the end-point of a string has been evaluated [31] to be consistent with the subleading term in the strong coupling expansion of the Bremsstrahlung function in [21].

Instead of the Nambu–Goto action we will use the string sigma model action in conformal gauge in the global coordinates to reconstruct the open string solution ending at the boundary which is associated with the two antiparallel Wilson lines in $S^3 \times R$ parametrized by two angle parameters ϕ and θ [18]. We will express the Minkowski open string solution associated with the BPS Wilson loop [15] specified by $\phi = \pm\theta$ in terms of the Poincaré coordinates. We will first make the flip of world sheet coordinates and secondly perform the T-duality along the boundary directions with the radial inversion $z \to 1/z$ to see what kind of string configuration appears and examine how the BPS condition $\phi = \pm\theta$ is encoded in the transformed string configuration.

We consider the classical open string solution in $AdS_5 \times S^5$ ending on the Wilson loop at the boundary which interpolates smoothly between the 1/2 BPS two antipodal lines and the coincident two antiparallel lines [18]. The Wilson loop for the $\mathcal{N} = 4$ SYM theory in $S^3 \times R$ is given by

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp\left[\oint \left(iA_\mu \dot{x}^\mu + \Phi_I \Theta^I |\dot{x}|\right) ds\right] \tag{1}$$

and characterized by two parameters ϕ and θ, where the loop is made of two lines separated by an angle $\pi - \phi$ along the big circle on S^3, and θ specifies the coupling to the scalars Φ_I. By describing the angle along the big circle by φ and the time by t we parametrize the two lines extending to the future and the past time directions as

$$t = s, \quad \varphi = \frac{\phi}{2}, \quad \Theta^1 = \cos\frac{\theta}{2}, \quad \Theta^2 = \sin\frac{\theta}{2},$$
$$t = -s', \quad \varphi = \pi - \frac{\phi}{2}, \quad \Theta^1 = \cos\frac{\theta}{2}, \quad \Theta^2 = -\sin\frac{\theta}{2}. \tag{2}$$

The general open string solution with two arbitrary angles ϕ and θ was constructed by using the Nambu–Goto action [18].

We rederive the open string solution associated with the BPS Wilson loop with $\phi = \pm\theta$ by using the conformal gauge for the string sigma model in the global coordinates. For a static open string in $AdS_3 \times S^1$ with metric

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2 + d\vartheta^2 \tag{3}$$

we make the following ansatz

$$t = \tau, \qquad \rho = \rho(\sigma), \qquad \varphi = \varphi(\sigma), \qquad \vartheta = \vartheta(\sigma) \tag{4}$$

with the Minkowski signature in the world sheet. The Virasoro constraint yields

$$-\cosh^2\rho + (\partial_\sigma\rho)^2 + \sinh^2\rho(\partial_\sigma\varphi)^2 + (\partial_\sigma\vartheta)^2 = 0, \tag{5}$$

which reads

$$\cosh^2\rho = (\partial_\sigma\varphi)^2\left[(\partial_\varphi\rho)^2 + \sinh^2\rho + (\partial_\varphi\vartheta)^2\right]. \tag{6}$$

We make a choice of sign as

$$\partial_\sigma\varphi = \frac{\cosh\rho}{\sqrt{(\partial_\varphi\rho)^2 + \sinh^2\rho + (\partial_\varphi\vartheta)^2}}. \tag{7}$$

The equation of motion for φ

$$\partial_\sigma\left(\sinh^2\rho\, \partial_\sigma\varphi\right) = 0 \tag{8}$$

gives one integral of motion p as

$$\sinh^2\rho\, \partial_\sigma\varphi = \frac{1}{p}, \tag{9}$$

which is expressed through (7) as

$$\frac{\sinh^2\rho\cosh\rho}{\sqrt{(\partial_\varphi\rho)^2 + \sinh^2\rho + (\partial_\varphi\vartheta)^2}} = \frac{1}{p}, \tag{10}$$

where we choose p as a positive parameter. The equation of motion for ϑ

$$\partial_\sigma^2\vartheta = 0 \tag{11}$$

gives a solution $\vartheta = J\sigma$ and the other integral of motion J defined by $\partial_\sigma\vartheta = J$ is described through (7) as

$$\frac{\partial_\varphi\vartheta\cosh\rho}{\sqrt{(\partial_\varphi\rho)^2 + \sinh^2\rho + (\partial_\varphi\vartheta)^2}} = J. \tag{12}$$

We combine (12) with (10) to have

$$\frac{\partial_\varphi\vartheta}{\sinh^2\rho} = pJ. \tag{13}$$

The remaining equation of motion for ρ is

$$\partial_\sigma^2\rho - \cosh\rho\sinh\rho - \cosh\rho\sinh\rho(\partial_\sigma\varphi)^2 = 0, \tag{14}$$

whose first integral is contained in (5). Indeed differentiating (5) with respect to σ we have

$$\partial_\sigma\rho\left(\partial_\sigma^2\rho - \cosh\rho\sinh\rho\right) + \partial_\sigma\varphi\left(\sinh^2\rho\,\partial_\sigma^2\varphi + \cosh\rho\sinh\rho\,\partial_\sigma\rho\,\partial_\sigma\varphi\right) + \partial_\sigma\vartheta\,\partial_\sigma^2\vartheta = 0, \tag{15}$$

which yields (14) through (8) and (11). The expressions such as (10), (12) and (13) were presented in Ref. [18] for the general $\phi \neq \theta$ case with two arbitrary parameters J and p where p is defined to have opposite sign from ours.

Here we restrict ourselves to the $J = \pm 1/p$ case, that is, the $\phi = \pm\theta$ case. Substitution of (13) into (12) generates a differential equation for ρ

$$(\partial_\varphi\rho)^2 = p^2\cosh^2\rho\sinh^4\rho - \sinh^4\rho - \sinh^2\rho, \tag{16}$$

which becomes through (9) to be

$$(\partial_\sigma\rho)^2 = \frac{\cosh^2\rho(p^2\cosh^2\rho - (p^2+1))}{p^2\sinh^2\rho}. \tag{17}$$

The turning point ρ_0 ($\rho_0 \leqslant \rho$) of open string is specified by $\cosh^2\rho_0 = (p^2+1)/p^2$. Thus we have dealt with the string sigma model action to recover the same relevant equations as constructed by using Nambu–Goto string action in Ref. [18]. Using the string sigma model for the open string in $AdS_3 \times S^3$ with a spin