Physics Letters B 728 (2014) 427–432

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Ginzburg–Landau phase diagram of QCD near chiral critical point – chiral defect lattice and solitonic pion condensate

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ARTICLE INFO

Article history:
Received 20 September 2013
Accepted 18 November 2013
Available online 23 November 2013
Editor: J.-P. Blaizot

Keywords:
Chiral symmetry breaking
Charged pion condensate
Inhomogeneous condensates

ABSTRACT

We investigate the influence of the isospin asymmetry on the phase structure of quark matter near the chiral critical point systematically using a generalized version of Ginzburg–Landau approach. The effect has proven to be so profound that it brings about not only a shift of the critical point but also a rich variety of phases in its neighborhood. In particular, there shows up a phase with spatially varying charged pion condensate which we name the "solitonic pion condensate" in addition to the "chiral defect lattice" where the chiral condensate is partially destructed by periodic placements of two-dimensional wall-like defects. Our results suggest that there may be an island of solitonic pion condensate in the low temperature and high density side of QCD phase diagram.

1. Introduction

The chiral critical point (CP) in QCD phase diagram is the subject of extensive theoretical/experimental studies [1]. It was shown in [2,3] that once the possibility of inhomogeneity is taken into account, the CP turns into a Lifshitz critical point (LCP) where a line of the chiral crossover meets two lines of second-order phase transitions surrounding the phase of an inhomogeneous chiral condensate. The inhomogeneous state can be viewed as an ordered phase separation, produced via the compromise between quark–antiquark attraction and a pair breaking due to imbalanced population of quarks and antiquarks [4,5]. Such inhomogeneity appears rather commonly in a wide range of physics; the Abrikosov lattice [6] and the Fulde–Ferrell–Larkin–Ovchinnikov superconductors [7] are such examples.

In this Letter, we address the question what is the possible impact of the effect of an isospin asymmetry on the LCP. For bulk systems such as matter realized in compact stars, the flavor symmetry breaking is caused mainly by a neutrality constraint that should be imposed to prevent the diverging energy density. The effect leads to a rich variety of color superconducting phases at high density [8]. On the other hand, at large isospin density QCD vacuum develops a charged pion condensate (PC) as soon as $|\mu_{\rm I}| > m_\pi$ with m_π and $\mu_{\rm I}$ being the vacuum pion mass and the isospin chemical potential [9]. The PC has a rich physical content including a crossover from a Bose–Einstein condensate of pions to a superfluidity of the Bardeen–Cooper–Schrieffer type, and has been extensively studied using effective models [10].

We focus here how the neighborhood of CP is to be modified by inclusion of isospin density. To this aim, we use the generalized Ginzburg–Landau (GL) approach developed in [2,4] which can give rather model-independent predictions near the CP. Since we are interested in the response of the CP and its vicinity against $\mu_{\rm I} \neq 0$, our strategy is to take $\mu_{\rm I}$ as a perturbative field and expand the GL functional with respect to it. The inclusion of $\mu_{\rm I}$ further brings new GL parameters, but they can be evaluated within the quark loop approximation [2,4] since gluons are insensitive to isospin charge. What we will find is that the isospin asymmetry dramatically modifies the neighborhood of CP bringing about new multicritical points. Accordingly, an inhomogeneous version of charged pion condensate dominates a major part of phase diagram.

2. Generalized Ginzburg–Landau approach

We consider two-flavor QCD, and assume the existence of a tricritical point (TCP) in the (μ, T)-phase diagram in the chiral limit at vanishing $\mu_{\rm I}$. We take the chiral four vector $\phi = (\sigma, \boldsymbol{\pi}) \sim (\langle\bar{q}q\rangle, \langle\bar{q}i\gamma_5\boldsymbol{\tau}q\rangle)$ as a relevant order parameter of the system. A minimal GL description of TCP requires the expansion of the thermodynamic potential up to sixth order in ϕ. The resulting chiral O(4) invariant potential expanded up to the sixth order is, with incorporating the derivative terms as well [2,4]: $\omega[\phi(\mathbf{x})] = \sum_{n=1,2,3} \omega_{2n}[\phi(\mathbf{x})]$, where

$$\omega_2[\phi(\mathbf{x})] = \frac{\alpha_2}{2}\phi(\mathbf{x})^2, \quad \omega_4[\phi(\mathbf{x})] = \frac{\alpha_4}{4}\big(\phi^4 + (\nabla\phi)^2\big),$$

http://dx.doi.org/10.1016/j.physletb.2013.11.037

$$\omega_6[\phi(\mathbf{x})] = \frac{\alpha_6}{6}\bigg(\phi^6 + 3[\phi^2(\nabla\phi)^2 - (\phi,\nabla\phi)^2] + 5(\phi,\nabla\phi)^2 + \frac{1}{2}(\Delta\phi)^2\bigg). \tag{1}$$

The current quark mass adds to this a term $\omega_1[\sigma(\mathbf{x})] = -h\sigma(\mathbf{x})$ which explicitly breaks O(4) symmetry down to O(3), and thus makes the condensate align in the direction $\phi \to (\sigma, \mathbf{0})$. We use $\alpha_6^{-1/2}$ as a unit of an energy dimension. Accordingly we replace α_6 with 1, and every quantity is to be regarded as a dimensionless. Then via scaling $\phi \to \phi h^{1/5}$, $\mathbf{x} \to \mathbf{x}h^{-1/5}$ together with $\alpha_2 \to \alpha_2 h^{4/5}$, $\alpha_4 \to \alpha_4 h^{2/5}$, we can get rid of h in ω apart from a trivial overall scaling factor $h^{6/5}$, i.e., $\omega \to \omega h^{6/5}$. Then we set $h = 1$, and retain the original letters ϕ, $\mathbf{x}$, α_2, α_4 and ω hereafter, but we should keep in mind that they should scale as $h^{1/5}$, $h^{-1/5}$, $h^{4/5}$, $h^{2/5}$, $h^{6/5}$ respectively.

We assume that $\sigma(\mathbf{x})$ is spatially varying in one direction, z [2]. The Euler–Lagrange equation (EL), $\delta\omega/\delta\phi(z) = 0$, becomes

$$\begin{aligned} 6h &= \sigma^{(4)}(z) - 10\big(\sigma^2\sigma'' + \sigma\left(\sigma'\right)^2\big) - 3\alpha_4\sigma'' \\ &\quad + 6\sigma^5 + 6\alpha_4\sigma^3 + 6\alpha_2\sigma, \end{aligned} \tag{2}$$

where h is temporarily recovered to remind us that the term comes from the mass term. We try the ansatz [3]

$$\sigma(z) = A\,\mathrm{sn}(kz - b/2, \nu)\mathrm{sn}(kz + b/2, \nu) + B, \tag{3}$$

where "sn" is the Jacobi elliptic function with ν the elliptic modulus, and k, b, A, B are real parameters. We call the state the "chiral defect lattice" (CDL).[1] This is a spatially modulating state having a period $\ell_p = 2K(\nu)/k$. Let us first show that the ansatz actually provides a one-parameter family of solution to the EL (2) when suitable conditions for A, B, k and b are met. First, we note from (3), $\mathrm{sn}(kz,\nu)^2 = \frac{(\sigma(z)-B)/A+b_2}{1+\nu b_2(\sigma(z)-B)/A}$ with $b_2 \equiv \mathrm{sn}(b/2,\nu)$. The fact that $f(z) = \mathrm{sn}(kz,\nu)$ obeys the Jacobi differential equation $(f')^2 = k^2(1-f^2)(1-\nu^2 f^2)$ translates into

$$d_0 = \left(\sigma'\right)^2 + d_1\sigma + d_2\sigma^2 + d_3\sigma^3 + d_4\sigma^4, \tag{4}$$

where $\{d_0, d_1, d_2, d_3, d_4\}$ are functions of A, B, b, k and ν. We here give the expressions for d_3 and d_4 only,

$$d_3 = 4d_4\left(A\frac{\mathrm{cn}(b,\nu)\mathrm{dn}(b,\nu)}{\nu^2\mathrm{sn}^2(b,\nu)} - B\right), \quad d_4 = -\frac{k^2\nu^4\mathrm{sn}^2(b,\nu)}{A^2}. \tag{5}$$

Differentiating (4) with respect to z and dividing the result by $2\sigma'$, we obtain

$$-\frac{d_1}{2} = \sigma''(z) + d_2\sigma(z) + \frac{3d_3}{2}\sigma(z)^2 + 2d_4\sigma(z)^3. \tag{6}$$

Differentiating this twice we have

$$\begin{aligned} 0 &= \sigma^{(4)}(z) + 6d_4\sigma^2\sigma'' + 12d_4\sigma\left(\sigma'\right)^2 + d_2\sigma'' \\ &\quad + 3d_3\left(\sigma'\right)^2 + 3d_3\sigma\sigma''. \end{aligned} \tag{7}$$

Adding to this, $(f_0 + f_1\sigma(z))\times$(4) and $(g_0 + g_1\sigma(z) + g_2\sigma(z)^2)\times$(6) with f_0, f_1, g_0, g_1, g_2 being arbitrary constants, we obtain a wider fourth-order differential equation. Then by tuning $f_0 = g_1 = -3d_3$, we can get rid of unnecessary σ'^2 and $\sigma\sigma''$ terms, and setting $f_1 = -10 - 12d_4$, $g_2 = -10 - 6d_4$, $g_0 = -d_2 - 3\alpha_4$ leads to

$$\begin{aligned} \gamma\big(\{d_i\},\alpha_4\big) &= \sigma^{(4)}(z) - 10\big(\sigma^2\sigma'' + \sigma\left(\sigma'\right)^2\big) - 3\alpha_4\sigma'' \\ &\quad - 6d_4(5+4d_4)\sigma^5 - 5d_3(5+6d_4)\sigma^4 \\ &\quad + \sum_{n=3,2,1}\beta_n\big(\{d_i\},\alpha_4\big)\sigma^n, \end{aligned} \tag{8}$$

where γ and β_n $(n = 1,2,3)$ are simple algebraic functions of $d_0, d_1, d_2, d_3, d_4,$ and α_4. Matching the coefficients of σ^5 and σ^4 with those in (2) leaves two choices; $(d_3, d_4) = (0, -1)$ or $(0, -1/4)$. It turns out that the latter cannot satisfy the remaining constraints so we choose $(d_3, d_4) = (0, -1)$ which, with (5), constrains A and B as

$$A = k\nu^2\mathrm{sn}(b,\nu), \qquad B = k\frac{\mathrm{cn}(b,\nu)\mathrm{dn}(b,\nu)}{\mathrm{sn}(b,\nu)}. \tag{9}$$

The conditions $\beta_3 = 6\alpha_2$ and $\beta_2 = 0$ are then automatically satisfied, so we are left with two constraints $6h = \gamma$ and $6\alpha_2 = \beta_1$. Now that $\{d_i\}$ are functions of three variables $\{k, \nu, b\}$, the two conditions fix two of them, for instance, $\{k, b\}$ at a fixed elliptic modulus ν. Hence, the ansatz (3) together with (9) gives a one-parameter solution to (2). To our knowledge, this is the first demonstration of the fact that (3) constitutes a solution also in the GL functional approach which could be applied in a wide range of physics. The parameter ν is to be determined via the minimization of thermodynamic potential Ω, the spatial average of the energy density over one period $\ell_p = 2K(\nu)/k$:

$$\Omega(\nu;\alpha_2,\alpha_4) = \frac{1}{\ell_p}\int_{-\ell_p/2}^{\ell_p/2} dz\,\omega\big[\sigma(z)\big]. \tag{10}$$

Let us briefly check the two extreme limits, $\nu \to 1$ and $\nu \to 0$. First when $\nu \to 1$,

$$\sigma(z) \to \sigma_{\mathrm{sd}}(z) = \frac{k}{\mathrm{th}(b)}\big(1 - \mathrm{th}^2(b)f_{\mathrm{def.}}(kz,b)\big), \tag{11}$$

where the subscript "sd" refers to a "single-defect", and $f_{\mathrm{def.}}(kz,b) \equiv 1 - \mathrm{th}(kz+b/2)\mathrm{th}(kz-b/2)$. This describes a defect in chiral condensate, represented by a *soliton–antisoliton pair* located at $z = 0$. The homogeneous value gets eventually recovered as $|z| \to \infty$: $\sigma_{\mathrm{sd}}(\pm\infty) \equiv \sigma_L = k/\mathrm{th}(b)$. Since $k = \sigma_L\mathrm{th}(b)$, we regard $\sigma_{\mathrm{sd}}(z)$ as a function of z parametrized by σ_L and b. On the other hand, when $\nu \to 0$ the ansatz reduces to, retaining up to the first non-trivial order in ν,

$$\sigma(z) \to \sigma_{\sin}(z) = k\cot(b) - \nu^2\frac{k\sin(b)}{2}\cos(2kz). \tag{12}$$

This is the state where chiral condensate is about to develop a ripple sinusoidal wave on the homogeneous background. We denote the background chiral condensate as $k\cot(b) \equiv \sigma_S$, $\sigma_{\sin}(z)$ is now viewed as a function of z parametrized by σ_S, k, and vanishing ν.

3. Phase structure at $\mu_I = 0$

We compute the phase diagram via minimization of (10). The result is displayed in Fig. 1. The CP is indeed realized as the LCP where the three phases meet; the CDL phase with $\sigma(z)$, the chiral symmetry broken (χSB) phase with a homogeneous condensate σ_L, and the nearly symmetry-restored phase characterized by a smaller condensate σ_S. For illustration, also shown by a solid line is the line of would-be first-order transition. Fig. 2 shows how $\sigma(z)$ smoothly interpolates between σ_L and σ_S along $\alpha_4 = -4$. Displayed in the left panel is the max amplitude $\max_z[\sigma(z)]$ as a function of α_2. Abrupt drop in σ indicated by a solid line shows the location of would-be first-order transition which would have

[1] The ansatz is called the "solitonic chiral condensate" in [3]. As we will see later, the state can be viewed as periodically placed wall-like defects of chiral condensate, so we use the term "CDL" here.