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Relative information entropy in cosmology: The problem of information entanglement

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ABSTRACT

The necessary information to distinguish a local inhomogeneous mass density field from its spatial average on a compact domain of the universe can be measured by relative information entropy. The Kullback–Leibler (KL) formula arises very naturally in this context, however, it provides a very complicated way to compute the mutual information between spatially separated but causally connected regions of the universe in a realistic, inhomogeneous model. To circumvent this issue, by considering a parametric extension of the KL measure, we develop a simple model to describe the mutual information which is entangled via the gravitational field equations. We show that the Tsallis relative entropy can be a good approximation in the case of small inhomogeneities, and for measuring the independent relative information inside the domain, we propose the Rényi relative entropy formula.

1. Introduction

Understanding the evolution of local and global inhomogeneities of the universe is a central problem in modern cosmology. The question has been actively investigated from several different points of view (e.g. perturbation theory [1], exact models [2], numerical simulations [3]), and a more recent approach attacks from the direction of information theory [4,5]. In the work of Hosoya et al. it has been shown [4], that the Kullback–Leibler relative information entropy [6] – a standard notion in information theory – arises very naturally in this context. It can describe the “distinguishability” of an actual mass density field from its spatial average on a compact domain of the universe, and its time derivative can also provide the commutation relation between the time evolution and the volume average operation on the density field.

The KL entropy is therefore a very useful measure to study the evolution of cosmic inhomogeneities inside a given domain, however, since gravity is a long-range interaction, it is often the case that the mutual information between spatially separated but causally connected domains of the universe is also important to consider. In order to compute the entangled information via the gravitational field equations in a realistic, inhomogeneous model, one has to face a rather formidable problem. The exact way to do so would be to follow a dynamical volume partitioning method (see e.g. the work of Wiegand and Buchert [7]), where the matter distribution function would have to be known globally, by solving complicated system of the Einstein's field equations. From an observationally motivated point of view for example, this approach is particularly difficult.

To circumvent this issue, in this Letter, we propose a simple model where we consider a parametric extension of the KL measure to describe the evolution of cosmic inhomogeneities on a domain whose dynamics is informationally entangled to its causally connected surroundings. In order to extract the mutual information between separated domains, instead of computing the explicit relations between the distribution functions via the exact method above, we consider a different approach where the distribution functions are formally treated as independent, but the corresponding relative entropy functions are nonadditive. We show, that under some physically motivated assumptions, this approach leads quite generally to the Tsallis relative entropy formula [9], which can parametrically incorporate the long-range interaction property of the gravitational field, based on the causal structure of the problem. Although nonadditive (nonextensive) phenomena have been known in cosmology and gravitation theory for decades, and the standard Tsallis entropy [10] has also been investigated several times in cosmological applications [11], as far as we know, this is the first time that a similar approach is considered in connection with cosmic inhomogeneities and information entropy. For simplicity, in the present work we restrict our investigations to

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linearly perturbed, spatially flat, Friedmann–Lemaître–Robertson–Walker (FLRW) dust cosmologies, however we expect that our model can be extended to more general, inhomogeneous cosmological models as well. Throughout this Letter we use units such that $c = G = 1$.

2. Background

The KL relative information entropy for continuous probability distributions $p(x)$ and $\overline{p}(x)$ on a compact domain, D, is defined as

$$S_{KL}\{p|\overline{p}\} = \int_D p(x) \ln \frac{p(x)}{\overline{p}(x)} dx. \tag{1}$$

For studies of cosmic inhomogeneities, the relevant distributions to consider are the matter density $\rho(t, x^i)$ and its spatial average, $\overline{\rho}(t)$, over an arbitrary domain of the universe. Hosoya et al. [4] have shown, that for a dust continuum cosmological model, described by the metric

$$ds^2 = -dt^2 + g_{ik} dx^i dx^k, \tag{2}$$

with a predefined time-orthogonal foliation, the KL relative entropy can be written as

$$\frac{S_{KL}\{\varrho|\overline{\varrho}\}}{V_D} = \overline{\varrho \ln \frac{\varrho}{\overline{\varrho}}}, \tag{3}$$

where overbar denotes the volume average operation:

$$\overline{\psi}(t) = \frac{1}{V_D} \int_D \psi(t, x^i) \sqrt{g}\, d^3x, \tag{4}$$

defined for any scalar field ψ in the volume $V_D = \int_D \sqrt{g} d^3x$. In (4) g is the determinant of the 3-metric g_{ik}, and x^i are coordinates on a $t = \text{const.}$ hypersurface within a comoving gauge. By exploiting the identity $\dot{\overline{\psi}} - \overline{\dot{\psi}} = \overline{\psi\theta} - \overline{\psi}\,\overline{\theta}$ with $\theta = \dot{V}_D/V_D$ being the local expansion rate, and also applying the continuity equation $\dot{\overline{\varrho}} + \overline{\theta}\,\overline{\varrho} = \dot{\varrho} + \theta\varrho = 0$, it can be shown [4] that the KL relative entropy is the generating functional of the commutation relation between the volume average and the time evolution of the density field, in the sense that

$$-\frac{\dot{S}_{KL}\{\varrho|\overline{\varrho}\}}{V_D} = \dot{\overline{\varrho}} - \overline{\dot{\varrho}}\,. \tag{5}$$

Hosoya et al. analyzed this relation, and argued that after long enough time, and on sufficiently large scales of averaging, $S_{KL}\{\varrho|\overline{\varrho}\}$ is an increasing function of time, and hence a reasonable entropy description for the evolution of local inhomogeneities.

3. The causal structure of the problem

The KL relative information entropy is additive for factorizing probabilities. More precisely, $S_{KL}\{\mathcal{A} \otimes \mathcal{B}\} = S_{KL}\{\mathcal{A}\} + S_{KL}\{\mathcal{B}\}$, if $\mathcal{A}$ and $\mathcal{B}$ are two *independent* systems in the sense that the probability distributions $p(x)$ and $\overline{p}(x)$ of $\mathcal{A} \otimes \mathcal{B}$ *factorizes* into those of $\mathcal{A}$ and of $\mathcal{B}$. As a consequence, the *mutual information*, defined usually as

$$I(\mathcal{A}, \mathcal{B}) \;=\; S\{\mathcal{A}\} + S\{\mathcal{B}\} - S\{\mathcal{A} \otimes \mathcal{B}\}, \tag{6}$$

is zero for composite independent systems. The assumption of independence, however, is obviously not valid for gravitationally coupled systems, like those in cosmology, where the evolution of a density field on an arbitrary compact domain of the universe is always influenced by its dynamics on its neighboring, causally connected regions via the gravitational field equations. On Fig. 1, we schematically plotted the causal structure of the evolution for an FLRW model. Here $J^-(\mathcal{A})$ is the causal past of $\mathcal{A}$ on an arbitrary reference time slice t_0 in the past, while α is the cosmological past of $\mathcal{A}$. $\mathcal{B}$ is a spatially separated domain from $\mathcal{A}$ on t_1, β is its cosmological-, while $J^-(\mathcal{B})$ is its causal past on t_0. It is clear from the picture that due to the long-range interaction property of the gravitational field (manifested by the intersection of $J^-(\mathcal{A})$ and $J^-(\mathcal{B})$), the mutual information $I(\mathcal{A}, \mathcal{B})$ at t_1 is obviously not zero, however as we have pointed out in the introduction, it is complicated to compute its value in general via the gravitational field equations in a realistic, inhomogeneous universe model by using an exact dynamical volume partitioning method, like the one in [7]. Nevertheless, since the time evolution of a density field on a domain D is clearly not independent from its evolution on causally connected regions to D, it would be nice to have a simple way to estimate the information entanglement between them.

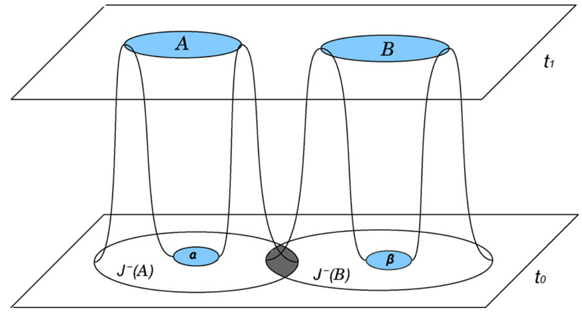

Fig. 1. The causal structure of the evolution of two spatially separated domains in an FLRW universe. α and β are the cosmological pasts of $\mathcal{A}$ and $\mathcal{B}$ respectively, while $J^-(\mathcal{A})$ and $J^-(\mathcal{B})$ are their intersecting causal pasts on t_0.

In the following sections, by considering a universe model described by a linearly perturbed FLRW metric with a dust continuum matter filed, we develop a parametric approximation for the entanglement problem. In this model we will stay completely inside the framework defined by Hosoya et al., i.e. we have a global, time orthogonal foliation with an inhomogeneous spacelike metric g_{ik} that is comoving with the matter perturbations in the considered linear order.

4. A parametric entropy extension

On large enough scales, as nicely confirmed by CMBR experiments [8], the universe can be well described in a thermodynamic equilibrium during its evolution. Unlike in standard thermodynamics however, as we have seen in the previous section, spatially separated domains do not evolve independently in general relativity, so it is a natural consequence that the entropy function of these regions are *not additive* for composition. When computing the joint entropy of separated domains in the model of Hosoya et al., the nonadditive part in the KL measure arises from the mutual dependence of the density and metric functions of the domains via the gravitational interaction. The practical computability of this entanglement is very difficult, and our approach in this Letter is to develop a toy model instead, where we formally treat the density functions as independent, but consider a parametric extension of the KL entropy function which is nonadditive even for independent distributions. The new entropy parameter will then be used to describe the mutual information between causally connected regions of the universe.

Based on the concept of composability alone, Abe showed [13], that the most general nonadditive entropy composition rule which is compatible with equilibrium requirements can be written in the form

$$H_\lambda(S_{\mathcal{A}\otimes\mathcal{B}}) = H_\lambda(S_\mathcal{A}) + H_\lambda(S_\mathcal{B}) + \lambda H_\lambda(S_\mathcal{A}) H_\lambda(S_\mathcal{B}), \tag{7}$$