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The superradiant instability regime of the spinning Kerr black hole

Shahar Hod [a,b,*]

[a] The Ruppin Academic Center, Emeq Hefer 40250, Israel
[b] The Hadassah Institute, Jerusalem 91010, Israel

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ABSTRACT

Spinning Kerr black holes are known to be superradiantly unstable to massive scalar perturbations. We here prove that the instability regime of the composed Kerr-black-hole-massive-scalar-field system is bounded from above by the dimensionless inequality $M\mu < m\cdot\sqrt{\frac{2(1+\gamma)(1-\sqrt{1-\gamma^2})-\gamma^2}{4\gamma^2}}$, where $\{\mu, m\}$ are respectively the proper mass and azimuthal harmonic index of the scalar field and $\gamma\equiv r_-/r_+$ is the dimensionless ratio between the horizon radii of the black hole. It is further shown that this *analytically* derived upper bound on the superradiant instability regime of the spinning Kerr black hole agrees with recent *numerical* computations of the instability resonance spectrum.

1. Introduction

The intriguing physical mechanism of superradiance [1–3] allows an incident bosonic wave field to extract rotational energy from a spinning Kerr black hole. In particular, a scalar field mode of azimuthal harmonic index m can be amplified (that is, can gain energy) as it scatters off a Kerr black hole if its proper frequency ω_{field} lies in the bounded regime [1–4]

$$0 < \omega_{\text{field}} < m\Omega_{\text{H}}, \tag{1}$$

where [5–7]

$$\Omega_{\text{H}} = \frac{a}{r_+^2 + a^2} \tag{2}$$

is the angular velocity of the spinning Kerr black hole (here a and r_+ are respectively the angular momentum per unit mass and the outer horizon-radius of the Kerr black hole).

What is even more remarkable is the fact that the rate of energy extraction from the spinning Kerr black hole in the superradiant regime (1) can grow exponentially in time if the scattered scalar wave field, which is used to extract the black-hole rotational energy, is prevented from radiating its energy to infinity. Interestingly, the Klein–Gordon wave equation for a scalar field of mass μ [8–11] in the Kerr black-hole spacetime is governed by an effective binding potential [see Eqs. (18) and (19) below] which provides a natural confinement mechanism that prevents low frequency field modes in the regime

$$0 < \omega_{\text{field}} < \mu \tag{3}$$

from escaping to infinity. Scalar field modes which respect the inequalities (1) and (3) in the rotating Kerr black-hole spacetime can grow exponentially over time [8], thus leading to the formation of a composed Kerr-black-hole-massive-scalar-field bomb [12,13].

The boundary between stable ($\omega > m\Omega_{\text{H}}$) and unstable ($\omega < m\Omega_{\text{H}}$) composed Kerr-black-hole-massive-scalar-field systems is marked by the presence of *stationary* field configurations whose orbital frequencies are in resonance with the angular velocity Ω_{H} of the spinning black hole [9,10]. Specifically, for a given value of the field azimuthal harmonic index m, these marginally-stable (stationary) bound-state field configurations are characterized by the resonance relation [9,10]

$$\omega_{\text{field}} = \omega_{\text{c}} \equiv m\Omega_{\text{H}}, \tag{4}$$

where ω_{c} is the critical (threshold) frequency for superradiant scattering in the Kerr black-hole spacetime.

It was previously proved [14,15] that, for a scalar field of proper mass μ interacting with a spinning Kerr black hole of angular velocity Ω_{H}, the inequality

$$\mu < \sqrt{2}\cdot m\Omega_{\text{H}} \tag{5}$$

provides an upper bound on the domain of existence of stationary Kerr-black-hole-massive-scalar-field configurations. Since these stationary (marginally-stable) field configurations mark the boundary between stable and unstable Kerr-massive-scalar-field systems,

* Correspondence to: The Ruppin Academic Center, Emeq Hefer 40250, Israel.
E-mail address: shaharhod@gmail.com.

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the relation (5) also provides an upper bound on the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system.

The main goal of the present paper is to derive a *stronger* upper bound on the superradiant instability regime of the spinning Kerr black-hole spacetime [16]. In particular, below we shall show that the binding potential well, which is required in order to support the stationary (marginally-stable) scalar field configurations (4) in the rotating Kerr black-hole spacetime, exists only in a restricted regime $\mu/m\Omega_{\text{H}} < \mathcal{F}(\gamma)$ [17] of the black-hole-field physical parameters. Since this inequality sets an upper bound on the domain of existence of these marginally-stable (stationary [16]) field configurations in the rotating Kerr black-hole spacetime, it also sets an upper bound on the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system.

2. Description of the system

We shall study the dynamics of a massive scalar field Ψ which is linearly coupled to a spinning Kerr black hole. The black-hole spacetime is described by the line element [5,6]

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\big[adt - (r^2+a^2)d\phi\big]^2, \tag{6}$$

where (t, r, θ, ϕ) are the Boyer–Lindquist coordinates, $\{M, a\}$ are the mass and angular momentum per unit mass of the black hole, and

$$\Delta \equiv r^2 - 2Mr + a^2 \quad ; \quad \rho^2 \equiv r^2 + a^2\cos^2\theta. \tag{7}$$

The zeros of Δ,

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{8}$$

are the (outer and inner) horizon radii of the spinning black hole.

The dynamics of a linearized scalar field Ψ of proper mass μ in the black-hole spacetime is governed by the Klein–Gordon wave equation

$$(\nabla^\nu\nabla_\nu - \mu^2)\Psi = 0. \tag{9}$$

One can decompose the eigenfunction Ψ of the massive scalar field in the form [18]

$$\Psi(t, r; \omega, \theta, \phi) = \sum_{l,m} e^{im\phi} S_{lm}(\theta; m, a\sqrt{\mu^2-\omega^2}) R_{lm}(r; M, a, \mu, \omega) e^{-i\omega t}. \tag{10}$$

Substituting (10) into the Klein–Gordon wave equation (9), one finds that the angular function S_{lm} satisfies the angular equation [19–24]

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{dS_{lm}}{d\theta}\Big) + \Big[K_{lm} + a^2(\mu^2-\omega^2)\sin^2\theta - \frac{m^2}{\sin^2\theta}\Big]S_{lm} = 0. \tag{11}$$

Demanding the angular functions to be regular at the two poles $\theta = 0$ and $\theta = \pi$, one finds that the differential equation (11) is characterized by a discrete set $\{K_{lm}\}$ of angular eigenvalues (see [25–27] and references therein). Below we shall use the fact that the characteristic eigenvalues of the angular equation (11) are bounded from below by the relation [27,28]

$$K_{lm} \geq m^2 - a^2(\mu^2 - \omega^2). \tag{12}$$

The radial function R_{lm} satisfies the radial equation [19,20]

$$\Delta\frac{d}{dr}\Big(\Delta\frac{dR_{lm}}{dr}\Big) + \Big\{[\omega(r^2+a^2) - ma]^2 + \Delta[2ma\omega - \mu^2(r^2+a^2) - K_{lm}]\Big\}R_{lm} = 0. \tag{13}$$

It is worth noting that the angular eigenvalues $\{K_{lm}\}$ couple equation (13) for the radial eigenfunctions to equation (11) for the angular eigenfunctions [29]. The radial equation (13) should be supplemented by the physical boundary condition of purely ingoing waves (as measured by a comoving observer) at the horizon of the black hole [8–10]:

$$R_{lm} \sim e^{-i(\omega - m\Omega_{\text{H}})y} \quad \text{for} \quad r \to r_+ \quad (y \to -\infty), \tag{14}$$

where the radial coordinate y is determined by the relation $dy = (r^2/\Delta)dr$ [see Eq. (17) below]. In addition, the asymptotic (large-r) behavior [8–10]

$$R_{lm} \sim \frac{1}{r}e^{-\sqrt{\mu^2-\omega^2}\,y} \quad \text{for} \quad r \to \infty \quad (y \to \infty) \tag{15}$$

of the radial eigenfunction, together with the characteristic inequality (3), guarantee that the external bound-state configurations of the massive scalar fields are characterized by spatially decaying (bounded) radial eigenfunctions at asymptotic infinity.

3. The effective binding potential of the composed Kerr-black-hole-massive-scalar-field system

Our main goal is to obtain an upper bound on the domain of existence of the stationary (marginally-stable) Kerr-black-hole-massive-scalar-field configurations [16]. To this end, it proves useful to transform the radial equation (13) into a Schrödinger-like wave equation. Substituting

$$\psi = rR \tag{16}$$

and [30]

$$dy = \frac{r^2}{\Delta}dr \tag{17}$$

into the radial equation (13), one obtains the Schrödinger-like wave equation

$$\frac{d^2\psi}{dy^2} - V(y)\psi = 0, \tag{18}$$

where the effective potential which governs the radial equation (18) is given by

$$V = V(r; \omega, M, a, \mu, l, m) = \frac{2\Delta}{r^6}(Mr - a^2) + \frac{\Delta}{r^4}[K_{lm} - 2ma\omega + \mu^2(r^2+a^2)] - \frac{1}{r^4}[\omega(r^2+a^2) - ma]^2. \tag{19}$$

Note that this radial potential is characterized by the asymptotic properties [see Eqs. (2), (4), and (19)]

$$V(r = r_+; \omega = \omega_{\text{c}}, M, a, \mu, l, m) = 0 \tag{20}$$

and

$$V(r \to \infty; \omega = \omega_{\text{c}}, M, a, \mu, l, m) \to \mu^2 - \omega_{\text{c}}^2 > 0 \tag{21}$$

at the black-hole horizon and at spatial infinity, respectively.

In the next section we shall analyze the spatial properties of the effective radial potential (19) for the stationary [16] bound-state configurations of the massive scalar fields in the rotating