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Instability of sphaleron black holes in asymptotically anti-de Sitter space-time

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ABSTRACT

We prove that sphaleron black holes in $\mathfrak{su}(2)$ Einstein–Yang–Mills–Higgs theory with a Higgs doublet in four-dimensional, asymptotically anti-de Sitter space-time are unstable.

1. Introduction

There is now a zoo of soliton and hairy black hole solutions of Einstein–Yang–Mills (EYM) theory and its variants in both asymptotically flat and asymptotically anti-de Sitter (adS) space-times (see [1–4] for some reviews). In pure EYM in four space-time dimensions with gauge group $\mathfrak{su}(2)$, all nontrivial, asymptotically flat, soliton [5] and black hole solutions [6–9] are unstable [10–14]. With an appropriate choice of gauge, linear, spherically symmetric, perturbations of the metric and gauge field decouple into two sectors, with different properties under a parity transformation: an even-parity (or gravitational) sector and an odd-parity (or sphaleronic) sector [12]. In the odd-parity sector, instability of the solitons [15] and black holes [16] can be proven by an elegant method using a variational technique; this does not require knowledge of the details of the equilibrium solutions, just their global behaviour and the boundary conditions on the fields. The instability in the odd-parity sector is similar to that of the flat-space electroweak sphaleron in $\mathfrak{su}(2)$ Yang–Mills–Higgs (YMH) theory [17–22] (hence the moniker “sphaleronic sector”), leading to a sphaleron interpretation of the soliton solutions [15].

Given this analogy between the flat-space YMH sphaleron and solutions of EYM theory, it is interesting to study gravitating solitons and black holes in Einstein–Yang–Mills–Higgs (EYMH) theory. In four-dimensional asymptotically flat space-time, with gauge group $\mathfrak{su}(2)$, sphaleron-like solutions have a doublet-Higgs field in the fundamental representation of the gauge group. Static, spherically symmetric, soliton and black hole equilibrium solutions of EYMH were studied numerically in [23]. There are two families of solutions, both of which share features with the pure EYM solutions (to which they reduce when the Higgs coupling is turned off). Like the EYM solutions, both the solitons [24] and black holes [25,26] in EYMH have a sphaleron-like instability in the odd-parity sector of linear, spherically symmetric perturbations.

It is well-known that the properties of EYM solitons and black holes in asymptotically adS space-time are radically different from those of the corresponding solutions in asymptotically flat space-time. In particular, for $\mathfrak{su}(2)$ gauge group, there exist nontrivial EYM solitons and black holes which are stable under linear, spherically symmetric perturbations in both the odd-parity and even-parity sectors [27–29].

The following question then arises: does this existence of stable pure EYM solutions in adS extend to EYMH solitons and black holes? Numerical solutions of the $\mathfrak{su}(2)$ EYMH equations, with a doublet Higgs field, in four-dimensional asymptotically adS space-time, were found some time ago [30].[1] The solutions resemble the asymptotically flat EYMH solitons and black holes studied in [23]

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[1] Solutions of the $\mathfrak{su}(2)$ EYMH equations in adS with a triplet Higgs field in the adjoint representation of the gauge group have also been found [31,32].

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rather than the asymptotically adS pure EYM configurations. By a simple extension of the analysis in [24], it can be shown that the asymptotically adS EYMH solitons have an instability in the odd-parity sector [30] analogous to the instability of the corresponding asymptotically flat EYMH solitons. Given this result, and the similarity between the asymptotically flat and adS EYMH black holes, the authors of [30] conjecture that the black hole solutions will also be unstable, but do not provide a proof since the techniques used in [25,26] to prove the instability of the asymptotically flat EYMH black holes do not extend to the asymptotically adS case.

In this note we close this gap by presenting a proof of the instability of black holes in $\mathfrak{su}(2)$ EYMH theory with a doublet Higgs field in asymptotically adS space-time under odd-parity, linear, spherically symmetric perturbations. In section 2 we outline the equilibrium and perturbation equations satisfied by the sphaleron black holes, following [24–26]. Our instability proof is in section 3 followed by brief conclusions in section 4.

2. Static and perturbation equations for EYMH theory in adS

We consider EYMH theory in four-dimensional, asymptotically adS space-time. The gauge group is $\mathfrak{su}(2)$ and the doublet Higgs field is in the fundamental representation. We focus on spherically symmetric soliton and black hole configurations with metric

$$ds^2 = -N(t,r)S^2(t,r)dt^2 + N^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{1}$$

For the $\mathfrak{su}(2)$ gauge field, we employ the ansatz [23,24,30]

$$A = a_0(t,r)\tau_r dt + a_1(t,r)\tau_r dr + [\omega(t,r)+1]\left[-\tau_\varphi d\theta + \tau_\theta \sin\theta\, d\varphi\right] + \tilde{\omega}(t,r)\left[\tau_\theta d\theta + \tau_\varphi \sin\theta\, d\varphi\right], \tag{2}$$

where the τ_i are generators of the $\mathfrak{su}(2)$ gauge group in spherical coordinates (see, for example, Appendix A of [26]). The doublet Higgs field takes the form [23,24,30]

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_2 + i\psi_1 \\ \phi(t,r) - i\psi_3 \end{pmatrix}, \tag{3}$$

where

$$\boldsymbol{\psi} = \psi(t,r)\hat{\mathbf{r}}. \tag{4}$$

The Higgs potential is

$$V(\Phi) = \frac{\lambda}{4}\left(\Phi^\dagger\Phi - v^2\right)^2, \tag{5}$$

where λ and v are constants.

Spherically symmetric EYMH solitons and black holes are described by the real quantities N, S, a_0, a_1, $\tilde{\omega}$, ω, ϕ and ψ. For static equilibrium configurations, all these quantities are functions of the radial coordinate r only. Furthermore, in this case we have $a_0 = a_1 = \tilde{\omega} = \psi = 0$ [23,30]. The remaining nonzero matter field functions $\omega(r)$, $\phi(r)$ satisfy the following static field equations, which arise from the Yang–Mills and Higgs equations [23,30]:

$$N\omega'' + \frac{(NS)'}{S}\omega' = \frac{1}{r^2}\left(\omega^2 - 1\right)\omega + \frac{\phi^2}{4}(1+\omega), \tag{6a}$$

$$N\phi'' + \frac{(NS)'}{S}\phi' + \frac{2N}{r}\phi' = \frac{1}{2r^2}\phi(1+\omega)^2 + \lambda\phi\left(\phi^2 - v^2\right), \tag{6b}$$

where a prime $'$ denotes differentiation with respect to r. The derivatives of the metric functions N and S can be written in terms of the matter field functions using the Einstein equations; we shall not require these equations for our analysis.

The variational method we employ in the next section does not depend on the details of the equilibrium matter fields, but the boundary conditions they satisfy will be crucial. We consider only the space-time exterior to a regular, nonextremal event horizon at $r = r_h$, in a neighbourhood of which the field variables take the form [30]

$$N(r) = O(r - r_h), \qquad S(r) = S_h + O(r - r_h),$$
$$\omega(r) = \omega_h + O(r - r_h), \qquad \phi(r) = \phi_h + O(r - r_h), \tag{7}$$

where S_h, ω_h and ϕ_h are constants. As $r \to \infty$, the space-time metric (1) tends to that of pure adS space-time, so that

$$N(r) = \frac{r^2}{\ell^2} + 1 + O(r^{-1}), \qquad S(r) = 1 + o(r^{-1}), \tag{8}$$

where ℓ is the adS radius of curvature. The matter fields have a complicated power-law decay as $r \to \infty$ [30]:

$$\omega(r) = -1 + \frac{c_1}{r^{k_1}}, \qquad \phi(r) = \pm v + \frac{c_2}{r^{k_2}}, \tag{9}$$

where c_1 and c_2 are constants, and the powers k_1 and k_2 are given by [30]

$$k_1 = \frac{1}{2}\left(1 + \sqrt{1 + v^2\ell^2}\right), \quad k_2 = \frac{3}{2}\left(1 + \sqrt{1 + \frac{8\lambda v^2\ell^2}{9}}\right). \tag{10}$$

The boundary conditions (9) at infinity constrain the YMH matter fields to have their vacuum values (as happens in the asymptotically flat case [23]), in contrast to the boundary conditions at infinity for pure EYM in adS, which do not constrain the value of the gauge field function ω as $r \to \infty$. This indicates that the asymptotically adS EYMH solitons and black holes are more like their counterparts in asymptotically flat space-time than the pure EYM solutions in adS. In asymptotically flat space-time, the matter functions ω and ϕ have an exponential rather than power-law fall off as $r \to \infty$ [23], but this does not have a major effect on the equilibrium solutions.

We now consider linear, spherically symmetric, perturbations of the static equilibrium configurations. By a choice of gauge, the perturbation δa_0 can be set to vanish identically [24]. The remaining perturbations decouple into two sectors: the even-parity (gravitational) sector consists of the perturbations of the metric functions δN and δS, together with the matter perturbations $\delta\omega$ and $\delta\phi$; the odd-parity (sphaleronic) sector contains the perturbations δa_1, $\delta\tilde{\omega}$ and $\delta\psi$. We consider only the latter sector of perturbations. All perturbations depend on time t as well as the radial coordinate r.

The linear perturbation equations for the odd-parity sector are the same in the asymptotically adS case as they are for the asymptotically flat case [24,30]. Defining a vector of perturbations $\Psi(t,r)$ by

$$\Psi(t,r) = \left(\delta a_1, \delta\tilde{\omega}, \delta\psi\right)^T, \tag{11}$$

they take the form [24,26]

$$-\mathcal{A}\ddot{\Psi} = \mathcal{H}\Psi, \tag{12}$$

where a dot $\dot{}$ denotes differentiation with respect to time t. The perturbation equations (12) involve two operators on the space of perturbations Ψ, namely [24,26]