Physics Letters B 756 (2016) 400–404

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Spacetimes with vector distortion: Inflation from generalised Weyl geometry

Jose Beltrán Jiménez [a], Tomi S. Koivisto [b,*]

[a] CPT, Aix Marseille Université, UMR 7332, 13288 Marseille, France
[b] Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, 10691 Stockholm, Sweden

ARTICLE INFO

Article history:
Received 7 January 2016
Received in revised form 17 March 2016
Accepted 17 March 2016
Available online 22 March 2016
Editor: M. Trodden

ABSTRACT

Spacetime with general linear vector distortion is introduced. Thus, the torsion and the nonmetricity of the affine connection are assumed to be proportional to a vector field (and not its derivatives). The resulting two-parameter family of non-Riemannian geometries generalises the conformal Weyl geometry and some other interesting special cases. Taking into account the leading nonlinear correction to the Einstein–Hilbert action results uniquely in the one-parameter extension of the Starobinsky inflation known as the alpha-attractor. The most general quadratic curvature action introduces, in addition to the canonical vector kinetic term, novel ghost-free vector-tensor interactions.

1. Spacetime degrees of freedom

In Einstein's General theory of Relativity (GR), gravitation is interpreted as curving of spacetime geometry, and can be described solely in terms of a metric. In addition to a *metric* structure however, a manifold representing a physical spacetime must also be endowed with an *affine* structure that determines the parallel transport. Though they coincide in GR, a priori these structures are both mathematically and physically independent [19].

Technically this can be formulated simply as the statement that the spacetime connection $\hat{\nabla}$ need not be the Levi–Civita connection ∇ as GR postulates. The ∇ is determined entirely by the metric $g_{\mu\nu}$ as given by the Christoffel symbols,

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\beta\lambda,\alpha} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\gamma} \right) . \tag{1}$$

This is the unique connection that is covariantly conserved, $\nabla_\alpha g_{\mu\nu} = 0$ and symmetric, $\Gamma^\alpha_{[\beta\gamma]} = 0$. The metric has $D(D+1)/2$ components in a D-dimensional spacetime, whereas the connection has D^3 components which are, in principle, completely independent degrees of freedom. Out of the D^3 components, $D^2(D-1)/2$ reside in the antisymmetric part

$$T^\alpha{}_{\beta\gamma} \equiv \hat{\Gamma}^\alpha_{[\beta\gamma]} , \tag{2}$$

which is called *torsion*. The remaining $D^2(D+1)/2$ degrees of freedom are encoded in the *non-metricity* tensor

$$Q_{\alpha\mu\nu} \equiv \hat{\nabla}_\alpha g_{\mu\nu} . \tag{3}$$

The *distortion* $\hat{\Gamma}^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma}$ of the affine structure is the combined effect of the torsion and the nonmetricity,

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + K^\alpha{}_{\beta\gamma} + D^\alpha{}_{\beta\gamma} , \tag{4}$$

where the *contortion* and the *deflection* tensors are defined as

$$K^\alpha{}_{\beta\gamma} = T^\alpha{}_{\beta\gamma} - T_{\beta\gamma}{}^\alpha - T_{\gamma\beta}{}^\alpha , \tag{5}$$

$$D^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left(Q_{\lambda\beta\gamma} - Q_{\beta\gamma\lambda} - Q_{\gamma\beta\lambda} \right) , \tag{6}$$

respectively [12].

2. Generalising Weyl geometry

The profound idea of gauge symmetry was brought forth within a pioneering non-Riemannian extension of the GR framework due to Hermann Weyl [23]. In Weyl's geometry, the metric compatibility condition is abandoned (while maintaining a symmetric connection) in such a way that the nonmetricity $Q_{\mu\alpha\beta}$ of the connection $\hat{\nabla}$ is determined by a vector A_μ as follows:

$$Q_{\mu\alpha\beta} \equiv \hat{\nabla}_\mu g_{\alpha\beta} = -2 A_\mu g_{\alpha\beta} . \tag{7}$$

* Corresponding author.
E-mail addresses: jose.beltran@cpt.univ-mrs.fr (J. Beltrán Jiménez), tomi.koivisto@nordita.org (T.S. Koivisto).

http://dx.doi.org/10.1016/j.physletb.2016.03.047

Table 1
Spacetimes with linear vector distortion. The second column indicates the number of free parameters.

Geometry	#	Constraints
General	2	–
Riemann	0	$b_1 = b_2 = b_3 = 0$
Dilation (Weyl)	0	$2b_1 - b_2 = b_3 = 0$
Generalised Weyl	1	$2b_1 - b_2 = b_3$
No dilation	1	$b_2 = b_3$
Pure deflection	0	$b_2 = b_3 = 0$
SVN [3]	1	$b_3 = 0$
Polar contortion	0	$b_1 = b_2 = b_3$
(Axial contortion	0	$b_1 = b_2 = b_3 = 0,\ b_4 \neq 0$)

Thus, a gauge symmetry arises because this relation is invariant under the (local) conformal transformation of the metric $g_{\mu\nu} \to e^{2\Lambda(x)} g_{\mu\nu}$ when simultaneously the vector is transformed as $A_\mu \to A_\mu - \partial_\mu \Lambda(x)$. The connection coefficients of $\hat{\nabla}$ derived from (7) are

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \left(A^\alpha g_{\beta\gamma} - 2A_{(\beta}\delta^\alpha_{\gamma)} \right) , \tag{8}$$

where the first term represents again the Christoffel symbols (1) and the expression inside the brackets is the deflection tensor (6). The theory obtained by writing the Einstein–Hilbert action in Weyl geometry is a trivial extension as the vector field is non-dynamical, and theories defined by nonlinear functions of the Einstein–Hilbert term turn out to be equivalent to the Palatini-$f(\mathcal{R})$ models. More general (Gauss–Bonnet-type) curvature terms however can generate new dynamical, ghost-free vector-tensor theories [7], see also [11].

In this letter we propose a *linear vector distortion* that generalises the Weyl geometry (8). That is, we consider the most general connection that is determined linearly by a vector field A_μ without derivatives. The distortion is then given by 3 independent terms[1]:

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - b_1 A^\alpha g_{\beta\gamma} + b_2 \delta^\alpha_{(\beta} A_{\gamma)} + b_3 \delta^\alpha_{[\beta} A_{\gamma]} . \tag{9}$$

We see that the original Weyl connection (8) is recovered for $b_2 = 2b_1 = 2$ and $b_3 = 0$. One of the parameters in (9) can actually be absorbed into the normalization of the vector field, but we will leave the three of them to track the effects of each term in the following. The torsion (2) and the non-metricity (3) tensors for the vector distortion are, respectively,

$$Q_{\mu\alpha\beta} = (b_3 - b_2) A_\mu g_{\alpha\beta} + (2b_1 - b_2 - b_3) A_{(\alpha} g_{\beta)\mu} , \tag{10}$$

$$T^\alpha{}_{\beta\gamma} = b_3 \delta^\alpha_{[\beta} A_{\gamma]} . \tag{11}$$

Now b_1 and b_2 contribute only to deflection, while b_3 causes also contortion. The torsion-free limit of this geometry, given by $b_3 = 0$ but general b_1 and b_2, has been in fact considered earlier in Ref. [3] (for other investigations into the nonmetric sector, see e.g. [13,4]). Some other special cases are listed in the Table 1.

Amongst them is, as an example, the Weyl–Cartan spacetime that arises from adding torsion to the Weyl connection (8). A remarkable class of geometries is obtained if we set $b_3 = 2b_1 - b_2$ in (10) so that we also recover the Weyl non-metricity relation given in (7). In detail, given $b_3 = 2b_1 - b_2$, we have $\nabla_\mu g_{\alpha\beta} = 2(b_1 - b_2) A_\mu g_{\alpha\beta}$, which is invariant under the Weyl transformation $g_{\mu\nu} \to e^{2\Lambda(x)} g_{\mu\nu}$ and $A_\mu \to A_\mu + \partial_\mu \Lambda(x)/(b_1 - b_2)$. This presents a whole family of generalised Weyl geometries where the gauge connection of the conformal covariant derivative carries also torsion, as seen from (11). Thus, we can introduce the covariant derivative $D_\mu g_{\alpha\beta} \equiv \Big[\partial_\mu - 2(b_1 - b_2) A_\mu\Big] g_{\alpha\beta}$, in terms of which the connection can be expressed as

$$\hat{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} \Big(D_\alpha g_{\lambda\beta} + D_\beta g_{\alpha\lambda} - D_\lambda g_{\alpha\beta} \Big) + K^\mu{}_{\alpha\beta} , \tag{12}$$

the first piece respecting the conformal invariance, but the contortion,

$$K^\mu{}_{\alpha\beta} = (b_2 - 2b_1) \left(A^\mu g_{\alpha\beta} - \delta_\alpha{}^\mu A_\beta \right) , \tag{13}$$

in general breaking it, unless $2b_1 - b_2 = 0$ and, hence, the torsion vanishes. The Weyl connection (8) is thus the unique conformally invariant connection, but the invariance of the non-metricity relation can be retained in a more general Weyl–Cartan spacetime given a fixed b_3.

Let us return to generic spacetimes described by the connection (9). The Riemann curvature it generates is given as

$$\mathcal{R}_{\mu\nu\rho}{}^\alpha \equiv \partial_\nu \hat{\Gamma}^\alpha_{\mu\rho} - \partial_\mu \hat{\Gamma}^\alpha_{\nu\rho} + \hat{\Gamma}^\alpha_{\nu\lambda} \hat{\Gamma}^\lambda_{\mu\rho} - \hat{\Gamma}^\alpha_{\mu\lambda} \hat{\Gamma}^\nu_{\nu\rho} , \tag{14}$$

and the corresponding Ricci curvature is just $\mathcal{R}_{\mu\rho} \equiv \mathcal{R}_{\mu\alpha\rho}{}^\alpha$. To form the scalar (Ricci) curvature we finally need also the metric, $\mathcal{R} \equiv g^{\mu\nu} \mathcal{R}_{\mu\nu}$. We find that two extra terms appear due to the nontrivial vector geometry:

$$\mathcal{R} = R - \beta_1 A^2 + \beta_2 \nabla \cdot A , \tag{15}$$

with (setting $D = 4$ from now on)

$$\beta_1 \equiv -\frac{3}{4} \Big[4b_1^2 - 8b_1(b_2 + b_3) + (b_2 + b_3)^2 \Big] , \tag{16}$$

$$\beta_2 \equiv -\frac{3}{2} (2b_1 + b_2 + b_3) . \tag{17}$$

Because of the projective invariance of the Ricci scalar (or, in general, of the symmetric part of the Ricci tensor), b_2 and b_3 only enter in the combination $b_2 + b_3$. This is so because such a symmetry implies an invariance under the transformation $\hat{\Gamma}^\alpha_{\mu\nu} \to \hat{\Gamma}^\alpha_{\mu\nu} + \delta^\alpha_\mu \xi_\nu$, for an arbitrary vector ξ_ν. This implies that the terms b_2 and b_3 will give degenerate effects unless the underlying gravitational theory breaks the projective invariance.

3. $f(\mathcal{R})$ actions

From the result (15), we see that the pure Einstein–Hilbert action $\mathcal{L} = M_{pl}^2 \sqrt{-g} \mathcal{R}/2$ in a spacetime with the linear vector distortion is equivalent to GR because the last term is a total derivative and the field equations for the vector field[2] imply $A_\mu = 0$. In order to obtain nontrivial non-Riemannian dynamics, one needs to consider a more general than the pure Einstein–Hilbert form of the action.

A natural starting point is then to take into account higher order curvature corrections that are expected to become relevant at high energies. For this purpose, we will consider prototypical extension of the Einstein–Hilbert action by including an arbitrary dependence upon the Ricci curvature scalar:

$$\mathcal{S} = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g} \mathcal{L} , \quad \mathcal{L} = f(\mathcal{R}) . \tag{18}$$

[1] The axial contortion term $b_4 \epsilon^\alpha{}_{\beta\gamma\mu} \tilde{A}^\mu$ is excluded because it would require that the field $\tilde{A}_\mu$ was a pseudovector. Let us mention that adding such a piece would not affect our results as the actions considered in this letter would imply $\tilde{A}_\mu = 0$.

[2] The mass of the vector vanishes if $\beta_1 = 0$, and becomes tachyonic for all parameter combinations for which $\beta_1 < 0$. These conditions generalise the result found in the torsion-free case, [3], that in our notation states that the mass is non-tachyonic if $b_1 = 2(2 - \sqrt{3}) b_2 < b_1 < 2(2 + \sqrt{3}) b_2$ when $b_3 = 0$. These conditions can be relevant if one promotes the vector action into the Proca by obtaining the Maxwell term from quadratic curvature invariants as in Ref. [7].