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# The photon angular momentum controversy: Resolution of a conflict between laser optics and particle physics

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ABSTRACT

The claim some years ago, contrary to all textbooks, that the angular momentum of a photon (and gluon) can be split in a gauge-invariant way into an orbital and spin term, sparked a major controversy in the Particle Physics community, exacerbated by the realization that many different forms of the angular momentum operators are, in principle, possible. A further cause of upset was the realization that the gluon polarization in a nucleon, a supposedly physically meaningful quantity, corresponds only to the gauge-variant gluon spin derived from Noether's theorem, evaluated in a particular gauge. On the contrary, Laser Physicists have, for decades, been happily measuring physical quantities which correspond to photon orbital and spin angular momentum evaluated in a particular gauge. This paper reconciles the two points of view, and shows that it is the gauge invariant version of the canonical angular momentum which agrees with the results of a host of laser optics experiments.



A major controversy has raged in Particle Physics recently as to whether the angular momentum (AM) of a photon, and *a fortiori* a gluon, can be split into physically meaningful, i.e. measurable, spin and orbital parts. The combatants in this controversy (for access to the controversy literature see the reviews by Leader and Lorcé [1] and Wakamatsu [2]) seem, largely, to be unaware of the fact that Laser Physicists have been measuring the spin and orbital angular momentum of laser beams for decades! (for access to the laser literature see the reviews of Bliokh and Nori [3], Franke-Arnold, Allen and Padgett [4] and Allen, Padgett and Babiker [5]). My aim is to reconcile these apparently conflicting points of view. Throughout this paper, unless explicitly stated, I will be discussing only free fields.

I shall first consider QED, where $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{A}$ are field operators, and as is customary, employ rationalized Gaussian units. It is usually stated that the momentum density in the electromagnetic field (known, in QED, as the *Belinfante* version) is proportional to the Poynting vector, i.e.

$$\boldsymbol{P}_{\rm bel} = \int d^3x\, \boldsymbol{p}_{\rm bel}(x) \qquad \boldsymbol{p}_{\rm bel}(x) = \boldsymbol{E}\times\boldsymbol{B} \tag{1}$$

and it is therefore eminently reasonable that the AM should be given by

$$\boldsymbol{J}_{\rm bel} = \int d^3x\, \boldsymbol{j}_{\rm bel}(x), \tag{2}$$

where the Belinfante AM density is

$$\boldsymbol{j}_{\rm bel} = \boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}). \tag{3}$$

Although this expression has the structure of an orbital AM, *i.e.* $\boldsymbol{r}\times\boldsymbol{p}$, it is, in fact, the *total* photon angular momentum density. On the other hand, application of Noether's theorem to the rotationally invariant Lagrangian yields the *Canonical* version which has a spin plus orbital part

$$\boldsymbol{J}_{\rm can} = \int d^3x\, \boldsymbol{j}_{\rm can} = \int d^3x\,[\boldsymbol{l}_{\rm can} + \boldsymbol{s}_{\rm can}] \tag{4}$$

where the canonical densities are

$$\boldsymbol{s}_{\rm can} = \boldsymbol{E}\times\boldsymbol{A} \quad \text{and} \quad \boldsymbol{l}_{\rm can} = E^i(\boldsymbol{x}\times\nabla)A^i \tag{5}$$

but, clearly, each term is gauge non-invariant.

Textbooks have long stressed a "theorem" that such a split cannot be made gauge invariant. Hence the controversial reaction when Chen, Lu, Sun, Wang and Goldman [6] claimed that such a split *can* be made. They introduce fields $\boldsymbol{A}_{\rm pure}$ and $\boldsymbol{A}_{\rm phys}$, with

$$\boldsymbol{A} = \boldsymbol{A}_{\rm pure} + \boldsymbol{A}_{\rm phys} \tag{6}$$

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where

$$\nabla \times \boldsymbol{A}_{\text{pure}} = \boldsymbol{0}, \quad \text{and} \quad \nabla \cdot \boldsymbol{A}_{\text{phys}} = 0 \tag{7}$$

which are, of course, exactly the same fields as in the Helmholz decomposition into longitudinal and transverse components[1]

$$\boldsymbol{A}_{\text{pure}} \equiv \boldsymbol{A}_{\parallel} \qquad \boldsymbol{A}_{\text{phys}} \equiv \boldsymbol{A}_{\perp}. \tag{8}$$

Chen et al. then obtain

$$\boldsymbol{J}_{\text{chen}} = \underbrace{\int d^3x\, \boldsymbol{E} \times \boldsymbol{A}_{\perp}}_{\boldsymbol{S}_{\text{chen}}} + \underbrace{\int d^3x\, E^i (\boldsymbol{x} \times \nabla) A^i_{\perp}}_{\boldsymbol{L}_{\text{chen}}} \tag{9}$$

and since $\boldsymbol{A}_{\perp}$ and $\boldsymbol{E}$ are unaffected by gauge transformations, they appear to achieved the impossible. The explanation is that the "theorem" referred to above applies to *local* fields, whereas $\boldsymbol{A}_{\perp}$ is, in general, *non-local*. In fact

$$\boldsymbol{A}_{\perp}(x) = \boldsymbol{A}(x) - \frac{1}{4\pi} \nabla \int d^3x' \, \frac{\nabla' \cdot \boldsymbol{A}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{10}$$

In all three versions of AM just mentioned, the integrands differ by terms of the general form $\nabla \cdot \boldsymbol{f}$, where $\boldsymbol{f}$ is some function of the fields, so that the integrated versions differ by surface terms at infinity, and thus agree with each other if the fields vanish at infinity. For classical fields, to state that a field vanishes at infinity, is physically meaningful, but what does it mean to say an operator vanishes at infinity? This issue is rarely addressed in the literature on the angular momentum controversy and the most recent serious analysis of this question seems to be that of Lowdon [7], utilizing axiomatic field theory. I shall comment later on his conclusions.

Now the key question is: what is the physical relevance of the various $\boldsymbol{S}$ operators? Can they be considered as genuine spin operators for the electromagnetic field? A genuine spin operator should satisfy the following commutation relations (for an interacting theory these should only hold as ETCs i.e. as Equal Time Commutators)

$$[\, S^i \,,\, S^j \,] = i\hbar \epsilon^{ijk} S^k. \tag{11}$$

But to check these conditions, manifestly, one must know the fundamental commutation relations between the fields and their conjugate momenta i.e. the quantization conditions imposed when quantizing the original classical theory, yet to the best of my knowledge, with only one exception [8], none of the papers in the controversy actually state what fundamental commutation relations they are assuming. Thus the expressions alone for the operators $\boldsymbol{S}$ are insufficient.

Failure to emphasize the importance of the commutation relations in a gauge theory can lead to misleading conclusions. It must be remembered that the quantization of a gauge theory proceeds in three steps:

(1) One starts with a gauge-invariant *classical* Lagrangian.
(2) One chooses a gauge.
(3) One imposes quantization conditions which are compatible with the gauge choice.

I shall comment on just two cases. In covariant quantization (cq) [9–11], for example in the Fermi gauge, one takes

$$[\, \dot{A}^i(\boldsymbol{x}, t) \,,\, A^j(\boldsymbol{y}, t) \,] = -i\delta^{ij} \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{12}$$

and then the Hilbert space of photon states has an indefinite metric.

Quantizing in the Coulomb gauge one uses transverse quantization (tq) (see e.g. [12])

$$[\, \dot{A}^i(\boldsymbol{x}, t) \,,\, A^j(\boldsymbol{y}, t) \,] = -i\delta^{\perp}_{ij}(\boldsymbol{x} - \boldsymbol{y}) \tag{13}$$

$$\equiv -i \int \frac{d^3k}{(2\pi)^3} \left( \delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2} \right) e^{i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{y})} \tag{14}$$

and the Hilbert space is positive-semidefinite.

There is an important physical consequence of this difference in quantization procedures. Gauge transformations on field *operators* almost universally utilize *classical* functions i.e.

$$\boldsymbol{A}(x) \to \boldsymbol{A}(x) + \nabla \alpha(x) \tag{15}$$

where $\alpha(x)$ is a "c-number" function. Clearly this transformation cannot alter the commutators. Or, put another way, gauge transformations are canonical transformations and therefore are generated by unitary operators, which do not alter commutation relations. This means that one cannot go from say Canonically quantized QED to Coulomb gauge quantized QED via a gauge transformation. This point was emphasized by Lautrup [9], who explains that although the theories are physically identical at the classical level, it is necessary to demonstrate that the physical predictions, meaning scattering amplitudes and cross-sections, are the same in the different quantum versions. This is also stressed by Cohen-Tannoudji, Dupont-Roc and Grynberg [13] on the basis that also the Hilbert spaces of the different quantum versions are incompatible.

It is not difficult to show that the canonical $\boldsymbol{S}_{\text{can}}$ with covariant quantization i.e. $\boldsymbol{S}^{\text{cq}}_{\text{can}}$ satisfies Eq. (11) and so is a genuine spin operator. However it is not gauge invariant. I shall comment on this presently.

For the Chen et al. case, since we are dealing with free fields, the parallel component of the electric field is zero i.e. $\boldsymbol{E}_{\parallel} = 0$ so that $\boldsymbol{J}_{\text{chen}}$ becomes

$$\boldsymbol{J}_{\text{chen}} = \int d^3x\, \boldsymbol{E}_{\perp} \times \boldsymbol{A}_{\perp} + \int d^3x\, E^i_{\perp} (\boldsymbol{x} \times \nabla) A^i_{\perp}. \tag{16}$$

But this is exactly the expression for $\boldsymbol{J}$, first discussed in [13], and later studied in great detail, with transverse quantization, by van Enk and Nienhuis (vE–N) in their classic paper [14], which, together with [15], is often the basis for statements about spin and orbital angular momentum in Laser Optics i.e. one has

$$\boldsymbol{S}^{\text{tq}}_{\text{chen}} \equiv \boldsymbol{S}^{\text{tq}}_{\text{vE–N}} = \int d^3x\, \boldsymbol{E}_{\perp} \times \boldsymbol{A}_{\perp} \tag{17}$$

and

$$\boldsymbol{L}^{\text{tq}}_{\text{chen}} \equiv \boldsymbol{L}^{\text{tq}}_{\text{vE–N}} = \int d^3x\, E^i_{\perp} (\boldsymbol{x} \times \nabla) A^i_{\perp}. \tag{18}$$

Now it is clear that $\boldsymbol{S}^{\text{tq}}_{\text{vE–N}}$, and $\boldsymbol{L}^{\text{tq}}_{\text{vE–N}}$, which are gauge invariant, are exactly the same as the gauge-variant canonical versions evaluated *in the Coulomb gauge*. For this reason, following [1], we shall henceforth refer to the Chen et al. = van Enk–Nienhaus operators as the Gauge Invariant Canonical (gic) operators. Thus

$$\boldsymbol{J}_{\text{gic}} = \boldsymbol{L}_{\text{gic}} + \boldsymbol{S}_{\text{gic}} = \int d^3x\, [\boldsymbol{l}_{\text{gic}} + \boldsymbol{s}_{\text{gic}}] \tag{19}$$

where the densities are

$$\boldsymbol{s}_{\text{gic}} = \boldsymbol{E}_{\perp} \times \boldsymbol{A}_{\perp} \quad \text{and} \quad \boldsymbol{l}_{\text{gic}} = E^i_{\perp} (\boldsymbol{x} \times \nabla) A^i_{\perp}. \tag{20}$$

[1] Indeed the only reason for the new nomenclature was Chen et al.'s intention to extend these ideas to QCD.