Physics Letters B 755 (2016) 47–57

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Universal Racah matrices and adjoint knot polynomials: Arborescent knots

A. Mironov [a,b,c,d,*], A. Morozov [b,c,d]

[a] Lebedev Physics Institute, Moscow 119991, Russia
[b] ITEP, Moscow 117218, Russia
[c] Istitute for Information Transmission Problems, Moscow 127994, Russia
[d] National Research Nuclear University MEPhI, Moscow 115409, Russia

ARTICLE INFO

Article history:
Received 29 November 2015
Received in revised form 28 January 2016
Accepted 29 January 2016
Available online 2 February 2016
Editor: M. Cvetič

ABSTRACT

By now it is well established that the quantum dimensions of descendants of the adjoint representation can be described in a universal form, independent of a particular family of simple Lie algebras. The Rosso–Jones formula then implies a universal description of the adjoint knot polynomials for torus knots, which in particular unifies the HOMFLY (SU_N) and Kauffman (SO_N) polynomials. For E_8 the adjoint representation is also fundamental. We suggest to extend the universality from the dimensions to the Racah matrices and this immediately produces a unified description of the adjoint knot polynomials for all arborescent (double-fat) knots, including twist, 2-bridge and pretzel. Technically we develop together the universality and the "eigenvalue conjecture", which expresses the Racah and mixing matrices through the eigenvalues of the quantum $\mathcal{R}$-matrix, and for dealing with the adjoint polynomials one has to extend it to the previously unknown 6×6 case. The adjoint polynomials do not distinguish between mutants and therefore are not very efficient in knot theory, however, universal polynomials in higher representations can probably be better in this respect.

1. Introduction

Knot theory [1–5] is intimately connected to representation theory via the Reshetikhin–Turaev (RT) formalism [6–18]. For the simplest (torus) knots the Rosso–Jones formula [19–22] expresses knot invariants through just quantum dimensions and Casimir eigenvalues. For evaluating knot polynomials of other knots, one has to know the Racah matrices. In fact, for a broad family of arborescent (double-fat) knots [2,23,24] it is sufficient to know only two Racah matrices [11,13,14]. Moreover, in the case of self-contragredient representation they are proportional to each other so that one needs only one Racah matrix, and this is exactly the case of adjoint representation which we consider in this paper. However, in general also needed are the "mixing matrices" [8,17], which are contractions of several Racah matrices so that to construct them one has to know a series of Racah matrices.

A remarkable discovery in representation theory is *universality* [25–37], a possibility to describe quantities for all the simple Lie algebras by a universal formula, which is the same for unitary, orthogonal, symplectic and exceptional groups. It turns out that such universal description exists, provided one considers only adjoint representation and its descendants (representations, appearing in tensor powers of adjoint), instead of arbitrary descendants of the fundamental one.

Historically, the term "universality" refers to the notion of the "Universal Lie algebra", introduced by Vogel in [27], which, roughly speaking, was intended to describe the Λ-algebra of triple-ended Feynman diagrams (closely related to Connes–Kreimer description [40, 41]). These diagrams are related to the Vassiliev invariants and, on the physical side, to the perturbative expansion in Chern–Simons theory, and the universality came just as an observation, as it was also the case earlier [32]. We call some quantity in the theory of (quantum) simple Lie algebras universal, if it can be expressed as a smooth symmetric function of three parameters $u=q^\alpha,\ v=q^\beta,\ w=q^\gamma$, and takes values for a given simple Lie algebra at the corresponding points of Vogel's Table, see (15) below in this text (for the ADE case it appeared already in [25]).

* Corresponding author at: Lebedev Physics Institute, Moscow 119991, Russia.
E-mail addresses: mironov@lpi.ru, mironov@itep.ru (A. Mironov), morozov@itep.ru (A. Morozov).

http://dx.doi.org/10.1016/j.physletb.2016.01.063
0370-2693/

Recently in [38] it was suggested to extend the notion of universality to the adjoint knot polynomials, i.e. to express them in terms of three variables u, v, w instead of the more conventional two A, q. Conventional colored polynomials (in the "E_8-sector" of representation theory) appear on particular 2-dimensional slices of the (u, v, w) space. For instance, the $SU(N)$ (HOMFLY) and $SO(N)$ (Kauffman) polynomials are described by choosing $u = q^{-2}$, $v = q^2$, $A = q^N$ and $u = q^{-2}$, $v = q^4$, $A = q^{N-1}$ respectively. However, in terms of u, v, w the knot polynomials acquire an additional property: they are symmetric functions of these variables. In [38] such universal formulas were explicitly presented for a variety of knots that included 2- and 3-strand torus knots and links and the figure eight knot, for the 2-strand case they were later reproduced in [39]. Actually, for the torus case this is not a big surprise, because the Rosso–Jones formula providing the knot polynomials in this case does not contain anything but quantum dimensions and Casimir eigenvalues, which are known to possess universal description [27,29–31,33]. Still, the results of [38] were originally obtained without any use of the Rosso–Jones formula, moreover, in the figure eight case this formula is unapplicable at all. The actual message of [38], which we make explicit in the present and the sequel paper [42], is that *the universality can be lifted to the Racah and mixing matrices*, and thus all the knot polynomials in adjoint family can actually be represented in the universal form. In this paper we make this idea explicit for the entire family of arborescent knots (which can be presented by double-fat graphs), evaluation of their knot polynomials having been discussed in detail in [14].

We achieve this by developing another challenging conjecture: the eigenvalue conjecture (EC) of [10] (see also [43]) expressing the mixing matrices between $\mathcal{R}$-matrices acting on different pairs of adjacent strands in a braid through the eigenvalues of $\mathcal{R}$-matrices themselves (actually, those eigenvalues are $\lambda = q^\kappa$ with κ being value of the second Casimir operator).

Actually, the EC in [10] is formulated for the 3-strand closed braids, and then the mixing matrices are actually the Racah matrices. At the same time, once the Racah matrix is known, ideas of [14] can be used to evaluate the colored polynomials for the arborescent knots, which is a family very different from the 3-strand one (sometimes wider, sometimes narrower). In fact, the 3-strand family *per se* can also be studied by the method of the present paper, but this is a separate story that will be reported elsewhere [42]. Technically EC provides a solution to the Yang–Baxter equation

$$U\mathcal{R}U\mathcal{R}U = \text{diagonal} \tag{1}$$

with diagonal $\mathcal{R} = \text{diag}(\lambda_1, \ldots, \lambda_M)$ in the form of orthogonal matrix U,

$$U^{\text{tr}} = U, \qquad U^2 = I \tag{2}$$

with all entries U_{ij} explicitly expressed through the eigenvalues λ's (the number of Yang-Baxter equations is exactly equal to the number of independent angles in the orthogonal matrix). In [10] such solution was explicitly provided for $M \le 5$, for the purposes of the present paper we need an extension to $M = 6$, what is not at all straightforward (as emphasized also in [43]).

Note that in terms of representation theory, U is constructed as the Racah matrix that relates the two maps: $\underbrace{(V \otimes V)}_{Q} \otimes V \to W$ and $V \otimes \underbrace{(V \otimes V)}_{Q'} \to W$ (for the sake of simplicity, we discuss here only the case of knots, not links when one is not obliged to consider three coinciding representations V). This matrix is involved in evaluating the knot invariants from 3-strand braid representation. At the same time, evaluating the arborescent knot invariants involves the Racah matrices $\bar{S}$ and S that relate accordingly the maps $\underbrace{(V \otimes \bar{V})}_{Q} \otimes V \to V$ with $V \otimes \underbrace{(\bar{V} \otimes V)}_{Q'} \to V$ and $\underbrace{(\bar{V} \otimes V)}_{Q} \otimes V \to V$ with $\bar{V} \otimes \underbrace{(V \otimes V)}_{Q'} \to V$, where $\bar{V}$ denotes the representation contragredient to V. In the case of adjoint representation V, which is self-contragredient, all three matrices U, S and $\bar{S}$ are equal to each other and the property of one of them is immediately inherited by the remaining ones. In particular, since this case is multiplicity free, all three matrices are symmetric (due to symmetricity of $\bar{S}$), and their first row is (the property inherited from S, $\bar{S}$)

$$U_{1Q} = U_{\emptyset Q} = \frac{\sqrt{\mathcal{D}_Q}}{\mathcal{D}_{Adj}} \tag{3}$$

where $\mathcal{D}_Q$ denotes the quantum dimension of the representation Q.

What we do in the present paper, we solve two problems at once: we use the coincidence of these Racah matrices in order to apply the EC to U, additionally using its symmetricity and the form of the first row. This allows us to restore the matrix U and then we use $S = U$ to construct the universal adjoint polynomials of the arborescent knots.

The paper is organized as follows. In section 2 we construct a 6×6 mixing matrix that is further used in section 3 for constructing the universal adjoint knot polynomials of various arborescent knots. Section 4 contains some comments of extension from the family of arborescent knots and also a discussion of properties of the polynomials obtained in the present paper. The concluding remarks are in section 5.

This paper is a the first paper of series of two papers devoted to evaluating the universal adjoint polynomials. The second paper [42] contains the results for more general knots that can be presented by "fingered 3-strand closed braids" [16,17].

2. 6 × 6 mixing matrix

2.1. Generality

In accordance with what is said in the Introduction, we need to construct the Racah matrix $U_{QQ'}$ that maps $\underbrace{(Adj \otimes Adj)}_{Q} \otimes Adj \to Adj$ and $Adj \otimes \underbrace{(Adj \otimes Adj)}_{Q'} \to Adj$. Since