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Entanglement renormalization and two dimensional string theory

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A B S T R A C T

The entanglement renormalization flow of a $(1+1)$ free boson is formulated as a path integral over some auxiliary scalar fields. The resulting effective theory for these fields amounts to the dilaton term of non-critical string theory in two spacetime dimensions. A connection between the scalar fields in the two theories is provided, allowing to acquire novel insights into how a theory of gravity emerges from the entanglement structure of another one without gravity.

1. Introduction

Currently, striking connections between the spacetime structure in gravitational theories and patterns of entanglement in dual quantum theories have emerged [1–5]. These incipient insights have been mostly understood in the framework of the AdS/CFT correspondence [6–8]. The holographic formula of the entanglement entropy [1] is a dazzling manifestation of these connections. It has been also noteworthy to observe how hyperbolic geometries come associated to the entanglement renormalization tensor networks (MERA) [9] used in numerical investigations of the ground states of quantum critical systems [10]. Using MERA and particularly its continuous version, cMERA [11], geometric descriptions of relevant states in field theories have been provided [12–14]. However, it has not been possible to establish if these geometrical representations correspond to solutions of any known theory of gravity.

Our objective in this Letter is to provide a simple example in which the cMERA representation of a free $(1+1)$ dimensional quantum field theory can be described in terms of the solutions of a gravity theory. As usual in physics, useful information can be gained by considering low-dimensional models. Here, we find that the cMERA representation of the ground state of a free massive boson amounts to a known solution of string theory in two spacetime dimensions. This theory, despite being the 'simplest' string theory, retains many interesting features of its more complex peers in higher dimensions and remarkably, it can be nonperturbatively formulated in terms of a model of nonrelativistic fermions via the $c=1$ matrix model [15].

2. Entanglement renormalization for QFT

The Multi-Scale Entanglement Renormalization Ansatz (MERA) [9,11] is a real-space renormalization group procedure on the quantum state which represents the wavefunction of the quantum system (usually in its ground state) at different length scales labelled by u. In MERA, $u=0$ usually refers to the state at short lengths (UV-state $|\Psi_{UV}\rangle$). In general, this state is highly entangled and acts as a starting reference point for the renormalization flow. MERA carries out a renormalization transformation at each length scale u in which, prior to coarse graining the effective degrees of freedom at that scale, the short range entanglement between them is unitarily removed through a *disentangler*. The procedure is applied an arbitrary number of times until the IR-state $|\Psi_{IR}\rangle$ is reached.[1]

The MERA flow can be implemented in a reverse way: starting from $|\Psi_{IR}\rangle$, it works by unitarily adding entanglement at each length scale until the correct $|\Psi_{UV}\rangle$ is generated. To fix the concept, let us generate the state $|\Psi(u)\rangle$ obtained by adding some amount of entanglement between left and right propagating modes of momentum $|k| \le \Lambda e^{-u}$ to the state $|\Psi_{IR}\rangle$,

$$|\Psi(u)\rangle = P\, e^{-i\int_{u_{IR}}^{u} d\hat{u}\,(\mathcal{K}(\hat{u})+\mathcal{D})}\,|\Psi_{IR}\rangle. \qquad (1)$$

The symbol P is a path ordering operator which allocates operators with bigger u to the right and Λ is a UV momentum cut-off.

[1] For massive theories, $|\Psi_{IR}\rangle$ is a completely unentangled state. In massless CFT, $|\Psi_{IR}\rangle$ amounts to the entangled vacuum of the theory.

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The operator $\mathcal{K}(\hat{u})$ creates a definite amount of entanglement at a given scale u and, in its most general form can be written as,

$$\mathcal{K}(\hat{u}) = \int d^d k\, \Gamma(k/\Lambda)\, g(\hat{u}, k)\, \mathcal{O}_k, \tag{2}$$

where $\mathcal{O}_k$ is an operator acting at the energy scale given by k and $\Gamma(x) = 1$ for $0 < x < 1$ and zero otherwise. The function $g(\hat{u}, k)$ depends on the state and the model that one deals with and represents the strength of the entangling process at a given scale. The operator $\mathcal{D}$ corresponds to coarse-graining [11,12]. To focus only in the entanglement flow along cMERA while avoiding the effects of the coarse graining process it is useful to rescale the cMERA states as,

$$|\widetilde{\Psi}(u)\rangle = e^{iu\,\mathcal{D}}\, |\Psi(u)\rangle = P\, e^{-i\int_{u_{IR}}^{u} d\hat{u}\, \widetilde{\mathcal{K}}(\hat{u})}\, |\Psi_{IR}\rangle. \tag{3}$$

Now, the *entangler* operator is given in the *interaction* picture,

$$\widetilde{\mathcal{K}}(\hat{u}) = e^{-i\hat{u}\mathcal{D}}\, \mathcal{K}(\hat{u})\, e^{i\hat{u}\mathcal{D}} = \int d^d k\, \Gamma(k\, e^{\hat{u}}/\Lambda)\, g(\hat{u}, k\, e^{\hat{u}})\, \widetilde{\mathcal{O}}_k, \tag{4}$$

with $\widetilde{\mathcal{O}}_k = e^{-i\hat{u}\mathcal{D}}\, \mathcal{O}_k\, e^{i\hat{u}\mathcal{D}} = e^{-d\hat{u}}\, \mathcal{O}_{k\, e^{\hat{u}}}$.

This Letter will consider the ground state of a $d = 1$ free bosonic theory with an action given by,

$$S = \int dt dx \left[(\partial_t \phi)^2 + (\partial_x \phi)^2 - m^2 \phi^2 \right]. \tag{5}$$

For this model, $\widetilde{\mathcal{K}}$ reads as [12],

$$\widetilde{\mathcal{K}}(\hat{u}) = -\frac{i}{2} \int dk\, \left(g_k(\hat{u})\, a_k^\dagger\, a_{-k}^\dagger - g_k(\hat{u})^*\, a_k\, a_{-k} \right), \tag{6}$$

with $g_k(\hat{u}) = \Gamma(ke^{\hat{u}}/\Lambda)\, g(\hat{u}, k)$. The operators $a_k^\dagger, a_k$ are defined as the creation and annihilation operators of a field mode with momentum k with respect to $|0\rangle$, the ground state of the theory at $u = 0$. The commutation relations are $\left[a_k, a_p^\dagger\right] = \delta(k - q)$, and zero otherwise. With this, the cMERA state $|\widetilde{\Psi}(u)\rangle$ amounts to the SU(1, 1)/U(1) generalized coherent state [16],

$$|\Phi\rangle = \mathcal{N}\, \exp\{-\frac{1}{2} \int dk\, \left[\Phi_k(u)\, K_+ - \overline{\Phi}_k(u)\, K_- \right]\}\, |0\rangle, \tag{7}$$

with $\Phi_k(u) = \int_0^u g_k(\hat{u})\, d\hat{u}$, $\overline{\Phi}_k(u) \equiv \Phi_k^*(u)$ and a normalization constant given by $\mathcal{N} = \exp\{-1/2 \int dk\, |\Phi_k(u)|^2\}$. The bilinear bosonic operators defined by

$$K_+ = a_k^\dagger\, a_{-k}^\dagger, \quad K_- = a_k\, a_{-k}, \tag{8}$$

together with $K_0 = \frac{1}{2}(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} + 1)$, satisfy the Lie algebra commutation relations of the group SU(1, 1)

$$[K_0,\, K_\pm] = \pm K_\pm \quad [K_-,\, K_+] = 2K_0, \tag{9}$$

and

$$K_-\, |\Phi\rangle = \Phi_k(u)\, |\Phi\rangle, \quad \langle\Phi|\, K_+ = \overline{\Phi}_k(u)\, \langle\Phi|. \tag{10}$$

From this point of view, the cMERA flow amounts to a sequential generation of a set of coherent states $|\,\Phi\rangle$ where the state $|0\rangle$ acts as the reference state.[2] This set of coherent states satisfy,

$$\int d\mu(\Phi)\, |\Phi\rangle\langle\Phi| = \mathbf{I}, \tag{11}$$

[2] We refer to [14] for an analysis of the differential generation of entanglement required to construct the set of cMERA coherent states (7).

where $d\mu(\Phi)$ is the SU(1, 1)-invariant Haar measure on SU(1, 1)/U(1). Furthermore, each one of these states are one-to-one corresponding to the points in the coset SU(1, 1)/U(1) manifold except for some singular points [17]. Namely, the states $|\Phi\rangle$ are embedded into a topologically nontrivial space corresponding to a 2-dimensional hyperbolic space. In other words, each cMERA state $|\Phi\rangle$ corresponds to a point on a two dimensional hyperbolic space. It may be argued that once provided a suitable measure of the distance between the states $|\Phi\rangle$, then a geometric description of the cMERA renormalization flow should correspond to the metric of a two dimensional AdS space [14]. More to be said about this point later in this work in which, we turn to ask whether the cMERA renormalization flow for the model (5) may be considered in terms of a concrete gravitational theory (see also [18]).

3. cMERA path integral and effective action

Here, we formulate cMERA as a path integral using the coherent state formalism. To this aim, we consider the amplitude

$$G(u_F, u_{IR}) = \langle\Phi_F\, |P\, \exp\{-i \int_{u_{IR}}^{u_F} d\hat{u}\, \widetilde{\mathcal{K}}(\hat{u})\}\, |\Phi_{IR}\rangle. \tag{12}$$

Recalling that $\partial_u\, \Phi_k(u) = g_k(u)$, then if one follows the standard procedure of dividing the renormalization scale interval $(u_F - u_{IR})$ into N intervals, each with $\epsilon = (u_F - u_{IR})/N$, then inserting the resolution of identity (11) at each interval point,[3] and finally letting N go to infinity while dropping $\mathcal{O}(\epsilon^2)$ terms, the amplitude (12) can be written as a formal generalized coherent state path integral,

$$G(u_F, u_{IR}) = \int d\mu(\Phi, \overline{\Phi})\, \exp\{i\, \mathcal{S}_{\rm eff}[\Phi, \overline{\Phi}]\}, \tag{13}$$

where

$$\mathcal{S}_{\rm eff}[\Phi, \overline{\Phi}] = -\int_{u_{IR}}^{u_F} du\, \left[\mathcal{L}[\Phi, \overline{\Phi}; u] + \langle\Phi\, |\, \widetilde{\mathcal{K}}(u)|\, \Phi\rangle \right],$$

$$\mathcal{L}[\Phi, \overline{\Phi}; u] = \frac{1}{2i} \int dk\, \left[\overline{\Phi}_k(u)\, \partial_u\, \Phi_k(u) - \Phi_k(u)\, \partial_u\, \overline{\Phi}_k(u) \right]. \tag{14}$$

We have explicitly dropped out the projection operators onto the initial and final states but it must be noted that the Euler–Lagrange equations derived from $\mathcal{S}_{\rm eff}[\Phi, \overline{\Phi}]$ are accompanied by the boundary conditions $\Phi_k(u_F) \equiv \Phi_k(u_N)$ and $\Phi_k(u_{IR}) \equiv \Phi_k(u_0)$ respectively. Regarding this, the effective action only contains two terms. The second term is tantamount to the matrix element of the entangler operator $\widetilde{\mathcal{K}}$ in the coherent state basis while the first term $\mathcal{L}[\Phi, \overline{\Phi}; u]$ is pure geometric; it is indeed a Berry phase that describes how the quantum entanglement is created along the cMERA flow. Using the expressions (6), (8) and (10), it can be shown that $\mathcal{L}[\Phi, \overline{\Phi}; u] = \langle\Phi\, |\widetilde{\mathcal{K}}(u)|\, \Phi\rangle$, so $\mathcal{S}_{\rm eff}[\Phi, \overline{\Phi}]$ totally accounts for the quantum fluctuations along the cMERA flow and can be written as,

$$\mathcal{S}_{\rm eff}[\Phi, \overline{\Phi}] = i \int_{u_{IR}}^{u_F} du\, dk\, \left[\overline{\Phi}_k(u)\, \partial_u\, \Phi_k(u) - \Phi_k(u)\, \partial_u\, \overline{\Phi}_k(u) \right]$$
$$= -2 \int_{u_{IR}}^{u_F} du\, dk\, \left[\overline{\Phi}_k(u)\, \partial_u\, \Phi_k(u) \right]. \tag{15}$$

[3] We also must note that the transition amplitude between two different coherent states (7) is given by $\langle\Phi'|\,\Phi\rangle = \exp\!\left[-1/2 \int dk \left(|\,\Phi_k(u')|^2 + |\,\Phi_k(u)|^2 - 2\, \overline{\Phi}_k(u')\, \Phi_k(u) \right)\right]$.