Logarithmic corrections to gravitational entropy and the null energy condition

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A B S T R A C T

Using a relation between the thermodynamics of local horizons and the null energy condition, we consider the effects of quantum corrections to the gravitational entropy. In particular, we find that the geometric form of the null energy condition is not affected by the inclusion of logarithmic corrections to the Bekenstein–Hawking entropy.

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1. Introduction

The null energy condition (NEC) requires that, for any null vector v^μ at any point in spacetime,

$$T_{\mu\nu}v^\mu v^\nu \geq 0 . \tag{1}$$

This condition plays a critical role in classical general relativity, being used in proofs of the singularity theorems [1] as well as in black hole thermodynamics [2]. Expressed in the form of (1), the NEC appears as a property of matter, since it is defined in terms of the matter energy–momentum tensor. But even in some classical limit, quantum field theory, our best framework for describing matter, does not appear to have a consistency requirement of the form of (1), at least when matter is considered in isolation without coupling to gravity.

A clue to the origin of the null energy condition comes from its role in general relativity. Instead of being a property purely of matter, perhaps the null energy condition is really a property of a combined theory of matter and gravity. In this context, Einstein's equations imply a different, though equivalent, form of the NEC,

$$R_{\mu\nu}v^\mu v^\nu \geq 0 , \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor. Written in this way, the NEC can be interpreted as a constraint on spacetime geometry, rather than as a constraint on matter. Indeed, it is this geometric form of the null energy condition, known as the Ricci or null convergence condition, that is ultimately invoked in gravitational theorems, because it is the Ricci tensor that appears in the Raychaudhuri equation.

Thus, if (2) could be derived directly, we could uphold all the gravitational theorems that otherwise rely on the so-far-unprovable (1).

Recently, it has been shown that precisely this condition can be derived from string theory [3], which, indeed, is a combined theory of matter and gravity; the NEC arises essentially as the spacetime interpretation of the Virasoro constraint on the worldsheet. But remarkably, the NEC in its geometric form, (2), can also be derived in a completely different way: as a consequence of the second law of thermodynamics [4]. Specifically, one assumes there exists some underlying microscopic theory obeying the laws of thermodynamics, from which classical gravity emerges via some coarse-graining procedure.

The notion that gravity is emergent is no longer controversial. For example, in the AdS/CFT correspondence, quantum gravity in anti-de Sitter space is described by a conformal field theory dual. The conformal field theory does not itself contain the graviton among its fundamental degrees of freedom so, from the dual point of view, gravity as well as an extra spatial dimension are emergent phenomena. In this approach, gravity emerges globally. However, in a remarkable paper [5], Jacobson considered a local version of holography. The idea was to assume, in keeping with the universality of horizon entropy, that gravitational entropy can be associated to "local Rindler horizons". Such local horizons exist everywhere because, in the vicinity of any point, spacetime is effectively flat and the local Minkowski space can be expressed in accelerating coordinates. Jacobson then found that Einstein's equations arise from the first law of thermodynamics, applied to a local Rindler horizon. In the same spirit, as shown in [4], the null energy condition, in the form of (2), arises from the second law of thermodynamics, applied locally.

The string and emergent gravity derivations both consider classical matter and gravity. The natural next question is to ask

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whether quantum effects lead to violations of the NEC. Indeed, it is known that the matter form of the NEC is violated when first order quantum effects are taken into account, e.g., by Casimir energy. Nevertheless, it is not clear that this indicates a violation in the Ricci convergence condition (2). To understand this, consider the semi-classical Einstein equations,

$$G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle \,, \tag{3}$$

which describe the backreaction of quantum fields on a classical background. The effect of the fluctuating quantum fields is captured by the renormalized expectation value of the energy–momentum tensor $\langle T_{\mu\nu} \rangle$ over a particular background. The relevance of $\langle T_{\mu\nu} \rangle$ to spacetime geometry relies on the validity of an equation of the form of (3), but we are not aware of any rigorous derivation of this equation as the semi-classical limit of a theory of both quantum matter and quantum geometry. Indeed, an equation which treats gravity classically but matter quantum-mechanically appears to be in some tension with the spirit of string theory in which matter and gravity are treated in a unified manner. In principle $\langle T_{\mu\nu} \rangle$ can be derived from an effective action $S_{\mathrm{eff}}(g_{\mu\nu})$ describing the quantum matter fields propagating on the background metric $g_{\mu\nu}$. In that case, generally one finds that $\langle T_{\mu\nu} \rangle$ will depend on higher-curvature terms (see, e.g., [6]). The field equations, therefore, will in general include higher-curvature corrections to Einstein's equations, severing the link between the NEC as a constraint on matter (1) and the NEC as a constraint on geometry (2). Thus a violation in (1) does not imply a violation in (2), and vice versa.

In this note, we take a different approach. Rather than calculating $\langle T_{\mu\nu} \rangle$, and then trying to determine its gravitational implications, the novel idea here is to directly determine $R_{\mu\nu} v^\mu v^\nu$ in the semi-classical theory. Specifically, we use the known form of the quantum-corrected version of the Bekenstein–Hawking entropy [7] to obtain the Ricci convergence condition. We find that, if we replace the Bekenstein–Hawking entropy of a horizon with its one-loop generalization and apply the second law of thermodynamics, we again arrive at exactly the Ricci convergence condition (2). Quantum corrections, at least of the type that contribute to the entropy, do not appear to alter the condition; if these were the only quantum corrections, then, for example, singularity theorems would continue to hold even in the semi-classical theory.

2. Time derivatives of entropy

In the emergent gravity paradigm, gravity emerges out of the coarse-graining of some more fundamental microscopic system. Holography suggests that the degrees of freedom of this system presumably live in one dimension less than the dimensionality of spacetime. In particular, we will assume that classical gravity in spacetime emerges in some suitable thermodynamic limit of the underlying system, such that the coarse-grained entropy of the dual system corresponds to the local Bekenstein–Hawking entropy of a null congruence, as defined in the next section. (We emphasize that the thermodynamic matter system we will have in mind will always be this dual system rather than any matter that may live within spacetime.) Little is known about this microscopic theory. Nevertheless, a few general remarks can be made. Here we will review the discussion in [4]. Consider a finite thermodynamic system and let S_{max} be its maximum coarse-grained entropy. Broadly, there are two kinds of thermal systems: those that are at thermodynamic equilibrium, and those that are approaching equilibrium. For systems already at equilibrium, $S = S_{\mathrm{max}}$, and $\dot{S}, \ddot{S} = 0$.

For systems approaching equilibrium, $S < S_{\mathrm{max}}$ and the second law says that $\dot{S} \geq 0$. We will also be interested in the second

derivative of the entropy. Now, since the entropy tends to a finite maximum value as it approaches thermal equilibrium, and since $\dot{S} \geq 0$, it seems intuitively reasonable that the time derivative of entropy will be a decreasing function of time, so that $\ddot{S} \leq 0$. This inequality holds for a great many systems of interest. For example, consider a gas diffusing in $3+1$ dimensions. Starting with an initial Gaussian density profile $\rho(r, 0) \sim e^{-r^2/2}$, the diffusion equation gives the density profile at later times:

$$\rho(r, \tau) = \frac{1}{2 \left(\pi \left(1 + 2D\tau\right)\right)^{3/2}} \exp \left(-\frac{r^2}{2 \left(1 + 2D\tau\right)}\right) \,, \tag{4}$$

where D is the diffusion constant. A straightforward calculation then yields the entropy:

$$S(\tau) = -\int dV \rho \ln \rho \sim \frac{3}{2} \ln(1 + 2D\tau) \,. \tag{5}$$

It is easy to verify that $\ddot{S} = -(2/3)\dot{S}^2$, from which we see that

$$S \geq 0, \quad \dot{S} \geq 0, \quad \ddot{S} \leq 0 \,, \tag{6}$$

at all times.

As another example, consider a system of N macrostates. Since classical phase space trajectories are typically chaotic, the dynamics can be described probabilistically after a few Lyapunov times. Assuming the ergodic hypothesis, the probability distribution can be taken to be uniform over all microstates. Then a short argument [4] yields

$$S(t) \approx S_{\mathrm{max}}(1 - e^{-kt}) \,, \tag{7}$$

where k is some constant with units of inverse time, and S_{max} is the maximum entropy of the system, corresponding to the largest macrostate. This form of the entropy also satisfies (6) for all times. In particular, $\ddot{S} < 0$.

In fact, the non-positivity of $\ddot{S}$ can be proven quite generally [9] using Onsager theory, at least for a very broad class of near-equilibrium systems approaching equilibrium. Near equilibrium, the time-derivative of a density fluctuation obeys a linear Onsager relation:

$$\delta\dot{\rho} = \hat{L}\delta\rho \,. \tag{8}$$

Onsager argued on rather general grounds that whenever certain time-reversibility properties are satisfied, the matrix $\hat{L}$ is symmetric [8]. From this one can show [9] that $\ddot{S} \leq 0$. That is, when $\hat{L}$ is symmetric, as is generally the case, then a sufficient condition for the non-positivity of $\ddot{S}$ is that the system be near equilibrium; for such systems (6) apply. This is a sufficient condition but, as the examples above illustrate, not a necessary one; $\ddot{S}$ can even be non-positive away from equilibrium. Henceforth, we will assume that conditions (6) are indeed satisfied by the microscopic dual theory of gravity.

3. Thermodynamic origin of the null energy condition

Let us now translate these conditions on entropy to spacetime. The idea in the emergent gravity paradigm is to associate an underlying non-gravitational system, presumably living in one dimension less, such that the coarse-grained entropy of the microscopic system accounts for the Bekenstein–Hawking entropy. Now, the universality of the Bekenstein–Hawking entropy formula, derived originally for black hole horizons, suggests that it really counts gravitational degrees of freedom. This led Jacobson to associate gravitational entropy *locally* to local Rindler horizons, thereby obtaining Einstein's equations from the Clausius relation. In a similar