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Duality and helicity: A symplectic viewpoint

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ABSTRACT

The theorem which says that helicity is the conserved quantity associated with the duality symmetry of the vacuum Maxwell equations is proved by viewing electromagnetism as an infinite dimensional symplectic system. In fact, it is shown that helicity is the moment map of duality acting as an SO(2) group of canonical transformations on the symplectic space of all solutions of the vacuum Maxwell equations.

1. Introduction

The usual electromagnetic action in the vacuum,[1]

$$S = -\frac{1}{4}\int_M F_{\mu\nu}F^{\mu\nu}\,d^4x\,, \tag{1.1}$$

suffers from well-known nevertheless inconvenient defects, namely the *non-invariance* of the Lagrange density under various symmetry transformations and the consequent non-symmetric form of its energy-momentum tensor, requiring to resort to various “improvements” [1,2].[2] In particular, while the vacuum Maxwell equations are invariant w.r.t. *duality transformations*,

$$F \mapsto \hat{F} = \cos\theta\,F + \sin\theta\,\star(F), \tag{1.2}$$

for any real θ (where $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ and $\star(F) = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}dx^\mu \wedge dx^\nu$ is the Hodge dual electromagnetic field strength), the Lagrange density in (1.1) is *not invariant*. The apparent contradiction can be resolved by observing that a duality rotation (1.2) changes the Lagrange density by a mere surface term. It is therefore a symmetry of the action [2,3] and generates therefore, according to the Noether theorem, a conserved quantity identified here as the optical *helicity* [4]. The proof given in [4] is rather laborious, though, due to the complicated behavior of the vector potential and the subsequent use of the Hertz vector — a rather subtle, non-gauge-invariant tool. The treatment in [3] is also quite involved.

Another proposition [2,5,7] is to embed the Maxwell theory into a manifestly duality-symmetric one for which Noether's theorem yields a seemingly different expression, namely,

$$\chi_{CS} = \frac{1}{2}\int_{\mathbb{R}^3}(\boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{C}\cdot\boldsymbol{E})\,d^3\boldsymbol{r} \tag{1.3}$$

à la Chern–Simons, where $\boldsymbol{A}$ and $\boldsymbol{C}$ are vector potentials for the magnetic and the electric fields, $\nabla\times\boldsymbol{A} = \boldsymbol{B}$ and $\nabla\times\boldsymbol{C} = -\boldsymbol{E}$, respectively. It is worth noting that the second term in Eq. # (14) of [4] and, respectively, in Eq. # (2.9) of [3], both represent the vector potential for the dual field strength — a fact not recognized by none of these authors. See [2,6,7] for comprehensive presentations.

In the first term in (1.3) we recognize the (magnetic) *helicity*, $\chi_{mag} = \frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{B}\,d^3\boldsymbol{r}$ widely studied in (magneto)hydrodynamics [8], where it measures the winding of magnetic lines of force and/or fluid vortex lines, respectively. It is worth stressing that the magnetic helicity alone is not a constant of the motion in general, and the clue leading to (1.3) is that its non-conservation,

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[1] Integration is performed over Minkowski spacetime, M, endowed with metric $g = g_{\mu\nu}\,dx^\mu dx^\nu$ of signature $(+,-,-,-)$. Let us stress that we will content ourselves with a special relativistic treatment of duality, although our main results spelled out in the next sections clearly hold true (with minor modifications) in a fixed gravitational background.

[2] We refer to, e.g., [13] for a geometric standpoint associated with the principle of general covariance, enabling us to circumvent these difficulties.

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$$\frac{d}{dt}\chi_{\text{mag}} = -\int_{\mathbb{R}^3} \boldsymbol{E}\cdot\boldsymbol{B}\, d^3\boldsymbol{r}, \tag{1.4}$$

is precisely compensated by that of the second term [6]. A remarkable fact is that (1.3) combines two *Chern–Simons invariants* [9], for both the electromagnetic and its dual field.

Duality and helicity have attracted considerable recent attention, namely in optics [2,7,10] and in heavy ion physics [11]. Our own interest stems from studying the helicity of semiclassical chiral particles [12].

In this Note we explain the duality and helicity from yet another viewpoint, which bypasses Lagrangians and gauge fixing altogether. Our clue is to view the set of solutions of electromagnetism as (an infinite-dimensional) *symplectic* space [14–16].

2. Electromagnetism in the symplectic framework

In the framework of Hamiltonian mechanics [14] one works with manifolds endowed with a closed two-form ω. If $\dim\ker(\omega)$ has constant but nonzero dimension, ω is called presymplectic; if its kernel is zero dimensional, it is called symplectic. In the physical applications we have in mind, we start with a manifold such that $(\mathcal{V},\omega)$ is presymplectic and is referred to as an *"evolution space"*, where the dynamics takes place. The characteristic leaves which integrate $\ker(\omega)$ are identified with the motions of the system. The quotient of $\mathcal{V}$ by the characteristic foliation of ω, namely $\mathcal{M} = \mathcal{V}/\ker(\omega)$, is therefore endowed with a symplectic two-form Ω, whose pull-back to $\mathcal{V}$ is ω. Then $(\mathcal{M},\Omega)$ is what has been called the *"space of motions"* in [14]. Crnković and Witten [15] call it the "true phase space".

The next ingredient is a Lie group G of canonical transformations, i.e., of diffeomorphisms of $\mathcal{V}$ preserving the two-form ω. Denote by $\mathfrak{g}$ the Lie algebra of G, and by $Z_{\mathcal{V}}$ the infinitesimal action (fundamental vector field) on $\mathcal{V}$ associated with $Z\in\mathfrak{g}$.

We thus have $L_{Z_{\mathcal{V}}}\omega = 0$ so that $\omega(Z_{\mathcal{V}},\,\cdot\,)$ is a closed one-form for all $Z\in\mathfrak{g}$. We now say that $J:\mathcal{V}\to\mathfrak{g}^*$ is a *moment map* for $(\mathcal{V},\omega,G)$ if the stronger condition

$$\omega(Z_{\mathcal{V}},\,\cdot\,) = -d(J\cdot Z) \tag{2.1}$$

holds for all $Z\in\mathfrak{g}$.[3]

If the equations of motion are given by $\ker(\omega)$, as it happens in the mechanics of finite dimensional systems [14] and, as we will prove below, also for Maxwell's electromagnetism, then J clearly descends to the space of motions, $\mathcal{M} = \mathcal{V}/\ker(\omega)$, as the *Noetherian quantity* associated with the symmetry group G: indeed $J\cdot Z$ is a *constant of the motion* for all $Z\in\mathfrak{g}$.

Below we boldly extend this framework to the infinite dimensional "manifold" $\mathcal{M}$ which consists of all *solutions* of the vacuum Maxwell equations modulo gauge transformations we endow with a *symplectic structure*.[4]

Let us show how all this comes about. Our first aim is to translate the usual variational approach into a symplectic language. The actual physical variable is the potential one-form $A = A_\mu\, dx^\mu$ locally defined by $F = dA$.[5] Then the variation of the action (1.1) with respect to a variation $\delta A = \delta A_\mu\, dx^\mu$ of the 4-potential is

$$\delta S = \int_M \left[\partial_\nu(F^{\mu\nu}\delta A_\mu) + (\partial_\mu F^{\mu\nu})\delta A_\nu\right] d^4x\,. \tag{2.2}$$

Assuming that the fields drop off sufficiently rapidly at infinity — or that the variations δA have compact support — the surface term can be dropped, allowing us to deduce the vacuum Maxwell equations $\partial_{[\mu}F_{\nu\rho]} = 0$ and $\partial_\mu F^{\mu\nu} = 0$, also written as

$$dF = 0 \qquad \text{and} \qquad d\star(F) = 0. \tag{2.3}$$

Denote by $\mathcal{V}$ the space of one-forms A of Minkowski space M whose associated field strength, $F = dA$, is a *solution* of (2.3). We contend that $\mathcal{V}$, which can be thought of as an infinite-dimensional manifold (affine space), is an "evolution space" for the Maxwell theory.

Firstly, a variation of a *solution*, δA, is a "tangent vector" to $\mathcal{V}$ at $A\in\mathcal{V}$ if $A+\delta A$ is still a solution of the field equations which vanishes at spatial infinity (as A does). Since the associated field strength is $F+\delta F$, where $\delta F = d(\delta A)$, it follows that δF also satisfies the Maxwell equations, $d(\delta F) = 0$ and $d\star(\delta F) = 0$.

Now, adapting Souriau's procedure in [14], Sec. 7, to field theory, we define a symplectic form on the space of all solutions of the linear system (2.3). To this end, we consider the action (1.1) by integrating over the domain $M' = [t_0,t_1]\times\Sigma\subset M$ defined by a Cauchy 3-surface Σ with *arbitrary dates* t_0 and $t_1\neq t_0$, where t is some given time-function. When F is a solution of the Maxwell equations, the variation vanishes, $\delta S = 0$, and therefore Eq. (2.2) boils down to

$$0 = \int_M \partial_\nu(F^{\mu\nu}\delta A_\mu)\, d^4x = \int_{\Sigma_1}\star(F(\delta A)) - \int_{\Sigma_0}\star(F(\delta A))\,,$$

where $\Sigma_i = \{t_i\}\times\Sigma$ for $i = 0,1$, implying that the integral does not depend on the choice of t_0 and t_1; the one-form[6]

$$\boxed{\alpha(\delta A) = \int_\Sigma \star(F(\delta A)) = -\int_\Sigma \star(F)\wedge\delta A} \tag{2.5}$$

is therefore well-defined; it is the *Cartan one-form*. The expression (2.5) represents the *flux* of the vector field $F(\delta A) = (F^{\mu\nu}\delta A_\mu)\partial_\nu$ across the Cauchy surface Σ. Calculating the exterior derivative, $\omega = d\alpha$, via $d\alpha(\delta A,\delta' A) = \delta(\alpha(\delta' A)) - \delta'(\alpha(\delta A)) - \alpha([\delta,\delta']A)$, we find

$$\boxed{\omega(\delta A,\delta' A) = \int_\Sigma \delta A\wedge\star(\delta' F) - \delta' A\wedge\star(\delta F).} \tag{2.6}$$

The two-form (2.6) corresponds *exactly* to that given by Eq. # (23) in [15].

From this point on, we do not use any Lagrangian; the starting point of all our subsequent investigations will be the two-form (2.6).

Let us now show that $(\mathcal{V},\omega)$ becomes a formal *presymplectic space*. To that end, let us compute its characteristic distribution. We thus must determine the kernel of ω, i.e., all variations δA of a solution $A\in\mathcal{V}$ such that $\omega(\delta A,\delta' A) = 0$ for all $\delta' A$, subject to the constraint $\delta'(d\star(F)) = 0$ to comply with the field equations. Using a Lagrange multiplier, f, we look for all solutions δA of

[3] For each point x of V, the quantity $J(x)$ belongs to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, and contracting with $Z\in\mathfrak{g}$ yields a function $x\mapsto J(x)\cdot Z$ on $\mathcal{V}$.

[4] A rigorous treatment of this infinite-dimensional differentiable structure would require the use of, e.g., diffeology [17], especially when dealing with differential forms on this "diffeological space".

[5] One-forms and vector fields are identified by lifting and lowering indices using the Minkowski metric.

[6] In a coordinate system where the metric is $g = dt^2 - d\boldsymbol{x}^2$ and Σ given by $t = \text{const}$, Eq. (2.5) reads

$$\alpha(\delta A) = \int F^{\mu\nu}\delta A_\mu\,\partial_\nu t\, d^3\boldsymbol{x}. \tag{2.4}$$