Physics Letters B 739 (2014) 279–284

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Jeans instability in classical and modified gravity

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ARTICLE INFO

Article history:
Received 3 July 2014
Received in revised form 2 November 2014
Accepted 5 November 2014
Available online 8 November 2014
Editor: J. Hisano

ABSTRACT

Gravitational instability in classical Jeans theory, General Relativity, and modified gravity is considered. The background density increase leads to a faster growth of perturbations in comparison with the standard theory. The transition to the Newtonian gauge in the case of coordinate dependent background metric functions is studied. For modified gravity a new high frequency stable solution is found.

1. Introduction

The instability of self-gravitating systems was first investigated by Jeans [1] in non-relativistic Newtonian gravity. It was extended to General Relativity (GR) by Lifshitz [2] and nowadays it is widely used in cosmology to study the rise of density perturbations in the expanding universe [3–6]. The original Jeans approach is based on the Poisson equation

$$\Delta\Phi = 4\pi G\varrho, \tag{1.1}$$

which is not satisfied in the zeroth order approximation, because the potential Φ is considered as a first order quantity, while the matter density $\varrho = \varrho_b + \delta\varrho$ includes zero (background) and first order terms. This problem is discussed in several textbooks, as e.g. in the aforementioned references [3–6]. It is also noted in an early paper indicated to us by an anonymous referee [7], where the author writes: "In (1.1) the density and pressure are supposed to be uniform throughout the gas. In fact, however, if gravitation is taken into account the equation of hydrodynamic equilibrium has no solution for a finite uniform mass."

To cure this shortcoming Mukhanov [4] suggested adding an antigravitating substance, such as e.g. vacuum-like energy, which would counterbalance the gravitational attraction of the background, so that Eq. (1.1) would be satisfied at zeroth order. Alternatively in Ref. [8] the authors assumed that the background density is zero, so that Eq. (1.1) becomes a relation between first order terms. This problem is absent in cosmology, where the zeroth order background equations are satisfied. They are the usual Friedmann equations in a homogeneous, isotropic universe, see for instance Refs. [3–7].

In this paper we take a different approach to the classical Jeans problem, assuming that Eq. (1.1) is valid for zeroth order terms so the solution of the equations of motion leads to time dependent background energy density and gravitational potential. These evolve with time in accordance following the equations of motion. The characteristic timescale of variation of these quantities is close to the Jeans time [both are essentially the gravitational time $t_g \sim (G\varrho)^{-1/2}$], so the development of the Jeans instability goes faster than in the standard theory, where such effect is not taken into account. This problem is studied in Section 2.

The treatment of the Jeans instability in General Relativity starts from the Einstein equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \equiv \widetilde{T}_{\mu\nu}. \tag{1.2}$$

These equations automatically include the equations of motion of matter, namely the continuity and Euler equations. On the other hand the equations of motion of matter can be equivalently obtained from the conditions of covariant conservation of the energy–momentum tensor

$$D_\mu T^\mu_\nu = 0, \tag{1.3}$$

where D_μ is the covariant derivative in the gravitational field under scrutiny. Usually, it is technically more difficult to derive the Euler and continuity equations from (1.2) because in this case one has to include the terms proportional to the square of the Christoffel symbols in the expression for the Ricci tensor.

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http://dx.doi.org/10.1016/j.physletb.2014.11.004

We consider the problem of gravitational instability for an initially spherically symmetric distribution of matter which generates a Schwarzschild-like background gravitational field. In contrast to the usually considered cosmological case, the background metric is not only a function of time but also a function of space coordinates. It leads to difficulties in imposing the Newtonian gauge condition. The problem of gauge fixing and the instability in a coordinate dependent background are studied in Section 3.

The final part of the paper, Section 4, is devoted to gravitational instability in $F(R)$ modified gravity theories. In cosmology this problem was considered in several works for different forms of $F(R)$, see e.g. Refs. [9–11]. We thank an anonymous referee for indicating several relevant papers [9] to us.

An analysis of the Jeans instability for stellar-like objects in modified gravity was performed in Refs. [12,8]. In these works a perturbative expansion of $F(R)$ was performed either around $R = 0$ or $R = R_c$, where R_c is the present cosmological curvature scalar. In our work we expand $F(R)$ around the curvature of the background metric R_m, which is typically much larger than R_c.

2. Jeans instability in Newtonian theory with space and time dependent background

We consider a spherically symmetric cloud of particles with initially vanishing pressure and velocities, and study the classical non-relativistic Jeans problem in Newtonian gravity. The essential equations are the well known Poisson, Euler, and continuity equations:

$$\begin{cases} \Delta\Phi = 4\pi G\varrho, & \text{(a)} \\ \partial_t(\varrho\mathbf{v}) + \varrho(\mathbf{v}\nabla)\mathbf{v} + \nabla P + \varrho\nabla\Phi = 0, & \text{(b)} \\ \partial_t\varrho + \nabla(\varrho\mathbf{v}) = 0. & \text{(c)} \end{cases} \tag{2.1}$$

It has been already mentioned in the Introduction that the problem with these equations, as described in the book by Zeldovich and Novikov [3], is that a time independent ϱ is not a solution to these equations. To avoid this problem the authors suggested studying solutions in the cosmological background, while Mukhanov [4] proposed to introduce some repulsive force. Instead we consider the time dependent problem taking as initial value a homogeneous distribution $\varrho = \text{const.}$ inside a sphere with radius r_m, while outside this sphere $\varrho = 0$. The initial values of particle velocities and pressure are taken to be zero and the potential Φ is supposed to be a solution of the Poisson equation (2.1)(a):

$$\Phi_0(r > r_m) = -MG/r, \qquad \Phi_0(r < r_m) = 2\pi G\varrho_0 r^2/3 + C_0, \tag{2.2}$$

where $C_0 = -2\pi G\varrho_0 r_m^2$ is chosen such that the potential is continuous (the value of C_0 is not important for us), and the total mass of the gravitating sphere is $M = 4\pi\varrho_0 r_m^3/3$.

In what follows we will be interested in the internal solution for $r < r_m$. Now we can find how the background quantities ϱ, v, and P evolve with time at small t. From Eq. (2.1)(b) it follows that:

$$v_1(r,t) = -\nabla\Phi_0 t = -4\pi G\varrho_0 rt/3. \tag{2.3}$$

From the continuity equation (2.1)(c) we find

$$\varrho_1 = \frac{2\pi}{3}G\varrho_0^2 t^2 \quad \text{or} \quad \varrho_b(t,r) = \varrho_0 + \varrho_1 = \varrho_0\left(1 + \frac{2\pi}{3}G\varrho_0 t^2\right). \tag{2.4}$$

It is interesting that ϱ rises with time but remains constant in space. Because of the homogeneity of ϱ the pressure remains zero, i.e. $P_1 = 0$.

The time variation of the background potential is found using (2.1)(a):

$$\Phi_b(r,t) = \Phi_0 + \Phi_1 = \frac{2\pi}{3}Gr^2\varrho_0\left(1 + \frac{2\pi}{3}G\varrho_0 t^2\right). \tag{2.5}$$

Now we can study the evolution of perturbations over this time-dependent background. We proceed as usual, writing $\varrho = \varrho_b(r,t) + \delta\varrho$, $\Phi = \Phi_b(r,t) + \delta\Phi$, $v = v_1(r,t) + \delta v$, and $\delta P = c_s^2\delta\varrho$, where c_s is the speed of sound. Here all δ-quantities are infinitesimal and are neglected beyond first order. In what follows we also neglect the products of small sub-one quantities (i.e. ϱ_1 etc) with δ's. This significantly simplifies the calculations, while of course the results do not change significantly. We find:

$$\begin{cases} \Delta(\delta\Phi) = 4\pi G\delta\varrho, & \text{(a)} \\ \partial_t\delta\mathbf{v} + \nabla\delta\Phi + \delta\varrho/\varrho_0\nabla\Phi_b + \nabla\delta P/\varrho_0 = 0, & \text{(b)} \\ \partial_t\delta\varrho + \varrho_0\nabla(\delta\mathbf{v}) = 0. & \text{(c)} \end{cases} \tag{2.6}$$

Eq. (2.6)(b) contains the term $(\delta\varrho/\varrho_0)\nabla\Phi_b$ which explicitly depends on the coordinate r through the background potential $|\nabla\Phi_b| = 4\pi Gr\varrho_0/3$. We estimate this term substituting instead of r its maximum value r_m. To see if this term is essential, let us take the Fourier transform of the last term in Eq. (2.6)(b):

$$\int \frac{d^3k}{(2\pi)^3}\frac{\nabla\delta P}{\varrho_0}e^{-i\lambda t + i\mathbf{kr}} \sim kc_s^2\frac{\delta\varrho(\lambda,\mathbf{k})}{\varrho_0}. \tag{2.7}$$

So we have to compare kc_s^2 with $4\pi G\varrho_0/3$. Evidently,

$$4\pi Gr_m\varrho_0/3 = \frac{r_g}{2r_m^2}, \tag{2.8}$$

where $r_g = 2GM$ is the gravitational (Schwarzschild) radius and M is the total mass of the spherical cloud under scrutiny. If k is of the order of the Jeans wave number:

$$k \sim k_J = \frac{\sqrt{4\pi G\varrho_0}}{c_s}, \tag{2.9}$$

we can neglect the r-dependent term $(\delta\varrho/\varrho_0)\nabla\Phi_b$ in comparison to $\nabla\delta P/\varrho_0$ for $c_s > \sqrt{2r_g/(3r_m)}$. There is quite a large volume of the parameter space where this condition is fulfilled.

Taking the Fourier transform of Eqs. (2.6)(a)–(c) and neglecting the r-dependent term we obtain the eigenvalue equation:

$$k^2\left(\lambda^2 - c_s^2k^2 + 4\pi G\varrho_0\right) = 0. \tag{2.10}$$

For small k we find the usual exponential Jeans instability:

$$\frac{\delta\varrho_{J1}}{\varrho_0} \sim \exp\left[t\left(4\pi G\varrho_0 - c_s^2k^2\right)^{1/2}\right]. \tag{2.11}$$

However, these small perturbations have the same characteristic rising time $\sim 1/(4\pi G\varrho_0)^{1/2}$ as that of the classical rise of ϱ_1. We can estimate the impact of the rising background energy density on the rise of perturbations making an adiabatic approximation, namely replacing the exponent in Eq. (2.11) with the integral:

$$\frac{\delta\varrho_{J2}}{\varrho_0} \sim \exp\left\{\int_0^t dt\left[4\pi G\varrho_b(t,r) - k^2c_s^2\right]^{1/2}\right\}. \tag{2.12}$$

where $\varrho_b(t,r)$ is given by Eq. (2.4).

Estimating the above integral for small k we find that the enhancement factor $\delta\varrho_{J2}/\delta\varrho_{J1}$ is equal to 1.027 after a time $t = t_{grav}$, where $t_{grav} = 1/\sqrt{4\pi G\varrho_0}$, while for $t = 2t_{grav}$ it is 1.23, for $t = 3t_{grav}$ it is 1.89, and for $t = 5t_{grav}$ it is 11.9. Note that to derive (2.11) and (2.12) we assumed that $t < t_{grav}$, so we should not treat these factors as numerically accurate; still, we can interpret them as an indication that the rise of fluctuations is indeed faster than in the usual Jeans scenario.