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Self-gravitating field configurations: The role of the energy–momentum trace

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ABSTRACT

Static spherically-symmetric matter distributions whose energy–momentum tensor is characterized by a non-negative trace are studied analytically within the framework of general relativity. We prove that such field configurations are necessarily highly relativistic objects. In particular, for matter fields with $T \geq \alpha \cdot \rho \geq 0$ (here T and ρ are respectively the trace of the energy–momentum tensor and the energy density of the fields, and α is a non-negative constant), we obtain the lower bound $\max_r\{2m(r)/r\} > (2+2\alpha)/(3+2\alpha)$ on the compactness (mass-to-radius ratio) of regular field configurations. In addition, we prove that these compact objects necessarily possess (at least) *two* photon-spheres, one of which exhibits *stable* trapping of null geodesics. The presence of stable photon-spheres in the corresponding curved spacetimes indicates that these compact objects may be nonlinearly unstable. We therefore conjecture that a negative trace of the energy–momentum tensor is a *necessary* condition for the existence of stable, soliton-like (regular) field configurations in general relativity.

1. Introduction

Nonlinear solitons have a long and broad history in science. These regular particle-like configurations find applications in many areas of physics, such as general relativity [1], string theory [2], condensed matter physics [3], nonlinear optics [4], and astrophysics [5].

Let us denote by T^{μ}_{ν} the energy–momentum tensor of the matter fields which compose a nonlinear static soliton. A simple argument [1,6] then reveals that, in flat space, the sum of the principal pressures, $\sum_i p_i$ (here $p_i = T^i_i$), cannot have a fixed sign throughout the body volume. This can be seen from the conservation law $\partial_j T^j_i = 0$, which implies that the spatial components of the energy–momentum tensor satisfy [1,6]

$$\int_{R^3} T_{ij} d^3x = 0. \tag{1}$$

The volume integral (1) has a simple physical meaning: it states that the total stresses must balance in a static matter distribution [1,6]. The relation (1) then implies that the sum of the principal pressures must switch signs somewhere inside the volume of the extended body. In particular, no regular static matter distributions exist with $\sum_i p_i > 0$ throughout the entire space. Such systems are of a purely repulsive nature and thus the force balance is impossible [1,6].

Although for purely repulsive matter fields (with $\sum_i p_i > 0$ throughout the body volume) in *flat* space the force balance is impossible, the situation may change in *curved* spacetimes (that is, in the presence of gravity). This fact is nicely demonstrated by the existence of globally regular particle-like solutions of the coupled Einstein–Yang–Mills field equations [7]. These non-linear solitons describe extended objects in which the repulsive nature of the matter field [8] is balanced by the attractive nature of gravity.

2. The trace of the energy–momentum tensor

We have seen that any flat-space static matter distribution must be characterized by the relation $\sum_i p_i < 0$ in some part of it. Denoting by $\rho > 0$ [9] the energy-density of the matter fields, one concludes that the trace of the energy–momentum tensor,

$$T \equiv -\rho + \sum_i p_i, \tag{2}$$

must also be negative in this part of the system volume. Thus, a negative trace of the energy–momentum tensor, at least in some

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part of the system volume, is a necessary condition for the existence of static regular matter distributions in flat spacetimes.

However, it should be emphasized that this conclusion no longer holds true in curved spacetimes. In particular, static soliton-like field configurations which are characterized by a non-negative trace (throughout the *entire* configuration's volume) do exist. The gravitating Einstein–Yang–Mills solitons [7], which are characterized by the identity $T = 0$, are a well-known example for such regular particle-like configurations.

The main goal of the present paper is to analyze, within the framework of general relativity, the physical properties of regular self-gravitating field configurations whose energy–momentum tensor is characterized by a non-negative trace [10]. The rest of the paper is organized as follows: In Section 3 we shall describe our physical system. In particular, we shall formulate the Einstein field equations in terms of the trace of the energy–momentum tensor. In Section 4 we shall prove that matter configurations which are characterized by a non-negative energy–momentum trace are necessarily highly relativistic objects. In particular, we shall derive a lower bound on the compactness (mass-to-radius ratio [11,12]) of these extended physical objects. In Section 5 we shall prove that the curved spacetime geometries which describe these self-gravitating objects necessarily possess (at least) two photon-spheres, compact hypersurfaces on which massless particles can follow null circular geodesics. We shall show that one of these photon-spheres exhibits stable trapping of the null circular geodesics. We conclude in Section 6 with a summary of the main results.

3. Description of the system

We study static spherically symmetric matter configurations in asymptotically flat spacetimes. The line element describing the spacetime geometry takes the following form in Schwarzschild coordinates [12–15]

$$ds^2 = -e^{-2\delta}\mu dt^2 + \mu^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3}$$

The metric functions $\delta(r)$ and $\mu(r)$ in (3) depend on the Schwarzschild areal coordinate r. Regularity of the matter configurations at the center requires

$$\mu(r \to 0) = 1 + O\left(r^2\right) \quad \text{and} \quad \delta(0) < \infty. \tag{4}$$

In addition, asymptotically flat spacetimes are characterized by

$$\mu(r \to \infty) \to 1 \quad \text{and} \quad \delta(r \to \infty) \to 0. \tag{5}$$

The fields that compose the matter configurations are characterized by an energy–momentum tensor T^μ_ν. The Einstein equations, $G^\mu_\nu = 8\pi T^\mu_\nu$, are given by [12,13,16]

$$\mu' = -8\pi r\rho + (1-\mu)/r, \tag{6}$$

and

$$\delta' = -4\pi r(\rho + p)/\mu, \tag{7}$$

where $T^t_t = -\rho$, $T^r_r = p$, and $T^\theta_\theta = T^\phi_\phi = p_T$ are respectively the energy density, the radial pressure, and the tangential pressure of the fields [11], and a prime denotes differentiation with respect to r.

The gravitational mass $m(r)$ contained within a sphere of radius r is given by [17]

$$m(r) = \int_0^r 4\pi x^2 \rho(x) dx. \tag{8}$$

For the total mass of the configuration to be finite, the energy density ρ should approach zero faster than r^{-3} at spatial infinity:

$$r^3\rho(r) \to 0 \quad \text{as}\ r \to \infty. \tag{9}$$

Substituting the Einstein field equations (6) and (7) into the conservation equation

$$T^\mu_{r;\mu} = 0, \tag{10}$$

one finds

$$p'(r) = \frac{1}{2\mu r}\left[\mathcal{N}(\rho + p) + 2\mu T - 8\mu p\right] \tag{11}$$

for the pressure gradient, where

$$T \equiv -\rho + p + 2p_T \tag{12}$$

is the trace of the energy–momentum tensor, and

$$\mathcal{N}(r) \equiv 3\mu - 1 - 8\pi r^2 p. \tag{13}$$

Below we shall analyze the spatial behavior of the pressure function $P(r) \equiv r^2 p(r)$, whose gradient is given by [see Eq. (11)]

$$P'(r) = \frac{r}{2\mu}\left[\mathcal{N}(\rho + p) + 2\mu T - 4\mu p\right]. \tag{14}$$

We shall assume that the matter fields satisfy the following conditions:

(1) The dominant energy condition [18]. This means that the energy density bounds the pressures:

$$\rho \geq |p|, |p_T| \geq 0. \tag{15}$$

(2) The trace of the energy–momentum tensor is non-negative. Specifically, we shall assume that the trace is bounded from below by

$$T \geq \alpha \cdot \rho \geq 0, \tag{16}$$

where $\alpha \geq 0$ is a constant. Note that Eqs. (12) and (15) imply $T \leq 2\rho$, an inequality which restricts the value of α to the regime [19,20]:

$$0 \leq \alpha \leq 2. \tag{17}$$

From Eqs. (12), (15), and (16) one also finds

$$p_T = \frac{1}{2}\left[T + (\rho - p)\right] \geq 0. \tag{18}$$

4. Lower bound on the compactness of the matter distributions

In the present section we shall derive a lower bound on the compactness, $\max_r\{2m(r)/r\}$, of the regular field configurations. To that end, we shall first analyze the behavior of the pressure function $P(r)$ in the asymptotic regimes $r \to 0$ and $r \to \infty$:

(1) From Eqs. (4) and (11) one finds $p'(r) = 2(p_T - p)/r$ as $r \to 0$. Regularity of $p(r)$ therefore requires $p(0) = p_T(0) \geq 0$ [see Eq. (18)], which implies [21]

$$P(r \to 0) \to 0^+. \tag{19}$$

(2) From Eqs. (5) and (14) one finds $P'(r) \simeq 2rp_T$ as $r \to \infty$, which implies [22]

$$P'(r \to \infty) \to 0^+. \tag{20}$$

In addition, from Eqs. (9) and (15) one learns that $p(r)$ should approach zero faster than r^{-3} at spatial infinity, which implies [23]