Physics Letters B 753 (2016) 65–68

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Hydrodynamics of the Polyakov line in SU(N_c) Yang–Mills

Yizhuang Liu [a], Piotr Warchoł [b], Ismail Zahed [a,*]

[a] Department of Physics and Astronomy, Stony Brook University, Stony Brook, NY 11794-3800, USA
[b] M. Smoluchowski Institute of Physics, Jagiellonian University, PL-30348 Krakow, Poland

ARTICLE INFO

Article history:
Received 2 October 2015
Received in revised form 26 November 2015
Accepted 27 November 2015
Available online 8 December 2015
Editor: A. Ringwald

ABSTRACT

We discuss a hydrodynamical description of the eigenvalues of the Polyakov line at large but finite N_c for Yang–Mills theory in even and odd space-time dimensions. The hydro-static solutions for the eigenvalue densities are shown to interpolate between a uniform distribution in the confined phase and a localized distribution in the de-confined phase. The resulting critical temperatures are in overall agreement with those measured on the lattice over a broad range of N_c, and are consistent with the string model results at $N_c = \infty$. The stochastic relaxation of the eigenvalues of the Polyakov line out of equilibrium is captured by a hydrodynamical instanton. An estimate of the probability of formation of a Z(N_c) bubble using a piece-wise sound wave is suggested.

1. Introduction

Lattice simulations of Yang–Mills theory in even and odd dimensions show that the confined phase is center symmetric [1,2]. At high temperature Yang–Mills theory is in a deconfined phase with broken center symmetry. The transition from a center symmetric to a center broken phase is non-perturbative and is the topic of intense numerical and effective model calculations [3] (and the references therein). Of particular interest are the semi-classical descriptions and matrix models.

In the semi-classical approximations, the confinement–deconfinement transition is understood as the breaking of instantons into a dense plasma of dyons in the confined phase and their reassembly into instanton molecules in the deconfined phase [4,5]. This mechanism is similar to the Berezinskii–Kosterlitz–Thouless transition in lower dimensions [6], and to the transition from insulators to superconductors in topological materials [7]. In matrix models, the Yang–Mills theory is simplified to the eigenvalues of the Polyakov line and an effective potential is used with parameters fitted to the bulk pressure to study such a transition [8,9], in the spirit of the strong coupling transition in the Gross–Witten model [10].

Matrix models for the Polyakov line share much in common with unitary matrix models in the general context of random matrix theory [11]. The canonical example is Dyson circular unitary ensemble and its analysis in terms of orthogonal polynomials or a one-component Coulomb plasma. The Dyson circular unitary ensemble relates to the one-dimensional Calogero–Sutherland model [12] which is an effective model for quantum Luttinger liquids.

A useful analysis of one-dimensional interacting electron systems relies on hydrodynamics which does not require an exact solution of the many-body problem. The method treats the system in the continuum limit as a fluid, and allows for the understanding of both small amplitude collective phenomena (phonons) as well as large amplitude effects (solitons, schocks) [13,14]. A reduction of the many-body Hamiltonian onto the hydrodynamical collective degrees of freedom makes use of the collective quantization method developed in the context of quantum field theory [15] and extended to problems in condensed matter physics [16].

In this letter we develop a hydrodynamical description of the gauge invariant eigenvalues of the Polyakov line for an SU(N_c) Yang–Mills theory at large but finite N_c. We will use it to derive the following new results: 1/ a hydrostatic solution for the eigenvalue density that interpolates between a confining (uniform) and de-confining (localized) phase; 2/ explicit critical temperatures for the Yang–Mills transitions in $1+2$ and $1+3$ dimensions; 3/ a hydrodynamical instanton for the density distribution that captures the stochastic relaxation of the eigenvalues of the Polyakov line; 4/ an estimate of the fugacity or probability to form a Z(N_c) bubble using a piece-wise sound-wave.

* Corresponding author.
E-mail addresses: yizhuang.liu@stonybrook.edu (Y. Liu), piotr.warchol@uj.edu.pl (P. Warchoł), ismail.zahed@stonybrook.edu (I. Zahed).

http://dx.doi.org/10.1016/j.physletb.2015.11.078

2. Polyakov line in 1 + 2 dimensions

The matrix model partition function for the eigenvalues of the Polyakov line for $SU(N_c)$ in $1+2$ dimensions was discussed in [8]. If we denote by $\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_{N_c}})$ with $\sum_i \theta_i = 0$ the gauge invariant eigenvalues of the Polyakov line, then [8]

$$Z[\alpha,\beta] = \int \prod_{i=1}^{N_c} d\theta_i \prod_{i<j}^{N_c} |z_{ij}|^{\beta(T)} e^{-\alpha(T)\sum_{i<j} V(|z_{ij}|)} \tag{1}$$

with $z_{ij} = z_i - z_j$ and $z_i = e^{i\theta_i}$. The perturbative potential $V(z_{ij})$ is center symmetric and quadratic in leading order or $V(|z_{ij}|) \approx |z_{ij}|^2$, with $\alpha(T) = T^2 V_2/2\pi$ and V_2 the spatial 2-volume [8]. The mass expansion of the one-loop determinant gives $\beta(T) = m_D^2 V_2/\pi$ [8]. The Debye mass is self-consistently defined as $m_D^2 = N_c g^2 T(\ln(T/m_D) + C)/2\pi$ [17] to tame all infra-red divergences, with $C \approx 1.3$ from lattice simulations [18,19].

(1) can be regarded as the normalization of the squared and real many-body wave-function $\Psi_0[z_i]$ which is the zero-mode solution to the Shrödinger equation $H_0\Psi_0 = 0$ with the self-adjoint squared Hamiltonian

$$H_0 \equiv \sum_{i=1}^{N_c} (-\partial_i + \mathbf{a}_i)(\partial_i + \mathbf{a}_i) \tag{2}$$

with $\partial_i \equiv \partial/\partial\theta_i$ and the pure gauge potential $\mathbf{a}_i \equiv \partial_i S$. Here $S[z] = -\ln\Psi_0[z]$ is half the energy in the defining partition function in (1). In (2) the mass parameter is $1/2$.

3. Hydrodynamics

We can use the collective coordinate method in [15] to re-write (2) in terms of the density of eigenvalues as a collective variable $\rho(\theta) = \sum_{i=1}^{N_c} \delta(\theta - \theta_i)$. For that, we re-define $H_0 \to H$ through a similarity transformation to re-absorb the diverging 2-body part induced by the Vandermond contribution $\Delta = \prod_{i<j} |z_{ij}|^{\beta(T)}$, i.e. $\Psi = \Psi_0/\sqrt{\Delta}$ and $\sqrt{\Delta} H = H_0 \sqrt{\Delta}$. Now H is of the general form discussed in [15] and is amenable after some algebra to

$$H = \int d\theta\, (\partial_\theta \pi \rho\, \partial_\theta \pi + \rho \mathbf{u}[\rho]) \tag{3}$$

with the potential-like contribution

$$\mathbf{u}[\rho] = \left(A(\theta) - \frac{\pi\beta(T)\rho_H}{2} + \frac{1}{2}\partial_\theta \ln\rho \right)^2 \equiv \mathbf{A}^2 \tag{4}$$

Here

$$A(\theta) = \frac{1}{2}\alpha(T) \int d\theta' \rho(\theta')\, \partial_\theta V\left(2\sin\left(\frac{\theta - \theta'}{2} \right) \right) \tag{5}$$

and ρ_H is the periodic Hilbert transform of ρ

$$[\rho]_H \equiv \rho_H(\theta) = \frac{1}{2}\frac{\mathrm{P}}{\pi} \int \rho(\theta') \cot\left(\frac{\theta - \theta'}{2} \right) \tag{6}$$

As conjugate pairs, $\pi(\theta)$ and $\rho(\theta)$ satisfy the equal-time commutation rule $[\pi(\theta), \rho(\theta')] = -i\left(\delta(\theta - \theta') - 1/2\pi\right)$. We identify the collective fluid velocity with $v = \partial_\theta \pi$ and re-write (3) in the more familiar hydrodynamical form

$$H \approx \int d\theta \rho(\theta) \left(v^2 + \mathbf{u}[\rho] \right) \approx \int d\theta \rho(\theta) |v + i\mathbf{A}|^2 \tag{7}$$

modulo ultra-local terms. The Heisenberg equation for ρ yields the current conservation law $\partial_t \rho = -2\partial_\theta(\rho v)$, and the Heisenberg equation for v gives the Euler equation

$$\partial_t v = i[H, v] = -\partial_\theta \left(v^2 + \mathbf{A}^2 - \partial_\theta \mathbf{A} - \mathbf{A}\partial_\theta \ln\rho + \pi\beta[\mathbf{A}\rho]_H - 2\alpha[\mathbf{A}\rho]_S \right) \tag{8}$$

with the sine-transform $[\mathbf{A}\rho]_S = \int \sin(\theta - \theta')\mathbf{A}(\theta')\rho(\theta')$. Note that all the relations hold for large but finite N_c.

4. Hydro-static solution

The static hydrodynamical density follows from the minimum of (6) with $v(\theta) = 0$,

$$\beta(T)\pi\rho_H(\theta) - \partial_\theta \ln\rho(\theta) = 2A(\theta) \tag{9}$$

To solve (9), we insert the leading quadratic contribution $A(\theta) \approx 2\alpha(T)\sin^2(\theta/2)$ in (9)

$$\rho\rho_H - a\partial_\theta \rho = bc_1 \rho \sin(\theta) \tag{10}$$

with $a \equiv 1/\pi\beta(T)$, $b \equiv 2\alpha(T)/\beta(T)$ and c_1 the first moment of the density or $\pi c_1 \equiv \int_0^{2\pi} \rho(\theta)\cos\theta d\theta$. Let $\rho_0 = N_c/2\pi$ be the uniform eigenvalue density and $\rho_1 = \rho - \rho_0$ its deviation. Consider the Cauchy transform

$$G(z) = \frac{1}{\pi i} \int_{\mathcal{C}} \frac{\rho_1(\eta)}{\eta - z} d\eta \tag{11}$$

with $\eta = e^{i\theta}$. The contour $\mathcal{C}$ is counter-clockwise along the unit circle. $G(z)$ is a holomorphic function in the complex z-plane. Let G^+ and G^- be its realization inside and outside $\mathcal{C}$ respectively, so that

$$G^\pm(z \to e^{i\theta}) = \pm\rho_1(\theta) + i\rho_H(\theta) \tag{12}$$

We now carry the Hilbert transform on both sides of (10). Setting $G(z) = G^+(z)$ and using $2[\rho_1\rho_H]_H = \rho_H^2 - \rho_1^2$, we have for (10)

$$\frac{1}{2}G^2 + (\rho_0 - \frac{1}{2}bc_1(z - z^{-1}))G + az\partial_z G = bc_1\rho_0 z + \frac{1}{2}bc_1^2 \tag{13}$$

on the boundary $\mathcal{C}$, thus within the circle. Here, we should require $G(z=0) = 0$ to ensure that ρ_1 integrates to zero.

$a \approx 1/V_2$ is subleading and will be dropped. Thus (13) is algebraic in $G(z)$. Since $\rho(\theta) = \rho_0 + \mathrm{Re}\, G^+(z = e^{i\theta})$, careful considerations of the singularity structures of the quadratic solutions to (13) yield (Θ is a step function)

$$\rho(\theta) = \sqrt{bc_1}(\cos\theta + 1)^{\frac{1}{2}}(\cos\theta - \cos\theta_0)^{\frac{1}{2}}\, \Theta(|\theta_0| - |\theta|) \tag{14}$$

The analytic properties of $G(z)$ fix $c_1/\rho_0 = 1 + (1 - 1/b)^{\frac{1}{2}}$ and θ_0 at $\cos\theta_0 = 1 - 2\rho_0/bc_1$. For $b < 1$ the non-uniform solution with $\rho_1 \neq 0$ is absent. For $b \gg 1$, $c_1 \to 2\rho_0$ and

$$\rho(\theta) \to \frac{N_c}{2\pi}\sqrt{8b - 4b^2\theta^2} \tag{15}$$

Therefore (14) interpolates between a uniform density distribution ρ_0 (confined phase) and a Wigner semi-circle (deconfined phase) with a transition at $b = 1$ or $T_c = m_D$. In $1+2$ dimensions the fundamental string tension is given to a good accuracy by $\sqrt{\sigma_1}/g^2 N_c = ((1 - 1/N_c^2)/8\pi)^{\frac{1}{2}}$ [22]. Thus the ratio in $1+2$ dimensions

$$\frac{T_c}{\sqrt{\sigma_1}} = \frac{C}{2\pi}\left(\frac{8\pi}{1 - 1/N_c^2} \right)^{\frac{1}{2}} \to \sqrt{\frac{2}{\pi}}\, C \tag{16}$$

with $C \approx 1.3$ [18,19]. In Fig. 1 we show the behavior of (16) (upper curve) versus N_c, in comparison to the numerical fit $T_c/\sqrt{\sigma_1} = 0.9026 + 0.880/N_c^2$ to the lattice results (lower curve) in [23]. Amusingly, (16) at large N_c is consistent with $\sqrt{3/\pi}$ in the string model [20].