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On selfdual spin-connections and asymptotic safety

U. Harst *, M. Reuter

Institute of Physics, University of Mainz, Staudingerweg 7, D-55099 Mainz, Germany

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ABSTRACT

We explore Euclidean quantum gravity using the tetrad field together with a selfdual or anti-selfdual spin-connection as the basic field variables. Setting up a functional renormalization group (RG) equation of a new type which is particularly suitable for the corresponding theory space we determine the non-perturbative RG flow within a two-parameter truncation suggested by the Holst action. We find that the (anti-)selfdual theory is likely to be asymptotically safe. The existing evidence for its non-perturbative renormalizability is comparable to that of Einstein–Cartan gravity without the selfduality condition.

1. Introduction

While it has been clear for several decades that besides the metric approach to General Relativity there exists an essentially equivalent description in terms of tetrads and spin-connections ("Einstein–Cartan gravity"), one of the important surprises uncovered by the work of Ashtekar [1–4] was that after a canonical transformation to new field variables the spin-connection may be chosen *selfdual* (or anti-selfdual). For Lorentzian signature and structure group O(1, 3), selfdual connections are unavoidably complex, which complicates their quantization. In the Euclidean case, they are real, however, and the condition of selfduality precisely halves the number of the connection's independent (real) components. In the generic case when the connection is not necessarily (anti-)selfdual, the Euclidean form of Ashtekar's theory can be obtained from the Holst action [5]; it depends on the tetrad $e_a{}^\mu$ and on the spin-connection $\omega^{ab}{}_\mu$ via its curvature $F^{ab}{}_{\mu\nu}$:

$$S_{\text{Ho}}[e,\omega] = -\frac{1}{16\pi G}\int \mathrm{d}^4x\, e\left[e_a{}^\mu e_b{}^\nu\left(F^{ab}{}_{\mu\nu} - \frac{1}{2\gamma}\varepsilon^{ab}{}_{cd}F^{cd}{}_{\mu\nu}\right) - 2\Lambda\right] \tag{1.1}$$

Besides the two terms known from the familiar first-order approach to general relativity, S_{Ho} contains a third one, containing the a priori arbitrary Immirzi parameter, γ, [6,7].

The case of $\gamma = \pm 1$ is special as the action S_{Ho} then depends only on one chirality of the spin connection $\omega^{ab}{}_\mu$, i.e. on its selfdual or anti-selfdual part w.r.t. the O(4)-indices. The (Euclidean!) duality operator being defined as $(\star)^{ab}{}_{cd} = \frac{1}{2}\varepsilon^{ab}{}_{cd}$ squares to unity, and objects with eigenvalue $+1$ are called selfdual, those with eigenvalue -1 anti-selfdual. We can define a projector on the (anti-)selfdual part of any antisymmetric second rank O(4)-tensor by $(P^\pm)^{ab}{}_{cd} = \frac{1}{4}(\delta^a_{[c}\delta^b_{d]} \pm \varepsilon^{ab}{}_{cd})$ and decompose it into a selfdual and anti-selfdual part, e.g. $F^{ab} = F^{(+)\,ab} + F^{(-)\,ab}$. In S_{Ho} the combination of curvature and Immirzi term in the case of $\gamma = \pm 1$ leaves us with

$$\frac{1}{2}\int \mathrm{d}^4x\, e\left[e_a{}^\mu e_b{}^\nu\left(F^{ab}{}_{\mu\nu} \mp \frac{1}{2}\varepsilon^{ab}{}_{cd}F^{cd}{}_{\mu\nu}\right)\right] = \int \mathrm{d}^4x\, e\left[e_a{}^\mu e_b{}^\nu\left(P^{\mp\,ab}{}_{cd}F^{cd}{}_{\mu\nu}\right)\right]. \tag{1.2}$$

One can show that the (anti-)selfdual part of the field strength tensor of a generic spin connection equals the field strength tensor of the (anti-)selfdual part of that spin connection: $(P^\pm F)^{ab}(\omega) = F^{(\pm)\,ab}(\omega) = F^{ab}(\omega^{(\pm)}) = F^{ab}(P^\pm\omega)$. As a result, there are only 12 (rather than the usual 24) independent components of the spin connection $\omega^{(+)}$ or $\omega^{(-)}$ on which the action really depends when $\gamma = \mp 1$. Thus $S_{\text{Ho}} \equiv S[e,\omega^\pm]$ for these special values of the Immirzi parameter, and this action leads to 12 equations of motion, that can be solved for $\omega^{(\pm)}(e)$, when the invertibility of the tetrad is assumed [8]. One can show that $\omega^{(\pm)}(e)$ is the (anti-)selfdual projection of the spin connection corresponding to the Levi-Civita connection, $\omega^{(\pm)}(e)^{ab}{}_\mu = (P^\pm\omega_{\text{LC}})^{ab}{}_\mu$. This spin connection necessarily gives rise to a spacetime with a non-vanishing torsion [9,10]. Nonetheless, substituted into the tetrad equations of motions we, again, arrive at Einstein's equation. Thus we find their solutions among the classical solutions of selfdual gravity, albeit formulated in a spacetime with a connection differing from the

* Corresponding author.
E-mail addresses: harst@thep.physik.uni-mainz.de (U. Harst), reuter@thep.physik.uni-mainz.de (M. Reuter).

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usual Levi-Civita connection. In classical gravity without fermionic matter [11–13] this difference cannot be observed [14,15].

In quantum-dynamical computations, in particular at the off-shell level, differences can, and do occur, however. For the case where non-selfdual connections and tetrads were chosen to serve as the basic field variables these differences were studied in Refs. [16–19] by means of a functional renormalization group equation (FRGE). Hereby the main emphasis was on the possibility that the theory might be non-perturbatively renormalizable along the lines of the Asymptotic Safety scenario [20–26]. A perturbative investigation was reported in [27,28]. Given the large number of theories classically equivalent to, or observationally indistinguishable from General Relativity [29,2,30–37] it is conceivable that there exist several inequivalent, asymptotically safe quantum gravity theories [38,17,39].

For the case $\gamma = \pm 1$, to be studied in the present paper, the Holst action comprises a theory of gravity in (anti-)selfdual variables that depends on less independent field components. Therefore, when we try to compute a path integral over this action for a general value of γ, we have to expect divergences in the limit $\gamma \to \pm 1$, as the integration over the other duality component will not be suppressed at all. In order to set up a FRGE for the (anti-)selfdual case we thus have to eliminate the redundant field components before the operator traces on the RHS of the flow equation are evaluated. It will turn out that this elimination is rather straightforward if we employ the particular decomposition of the fluctuation fields advocated in [19]. This way we are able to study in this paper the non-perturbative RG flow of gravity in selfdual variables for the first time.

2. Non-perturbative RG flow of selfdual gravity

The starting point of the present investigation is the FRGE-based analysis of (non-selfdual) Einstein–Cartan quantum gravity that was performed in Ref. [19]. In this analysis the effective average action was approximated by $\Gamma_k = \Gamma_{\text{Ho}\,k} + \Gamma_k^{\text{gf}} + \Gamma_k^{\text{gh}}$. Here $\Gamma_{\text{Ho}\,k}$ denotes the Holst action S_{Ho} with running parameters $(G_k, \Lambda_k, \gamma_k)$, while Γ_k^{gf} and Γ_k^{gh} are the gauge-fixing and Faddeev–Popov ghost terms corresponding to the diffeomorphism and $\text{O}(4)_{\text{loc}}$ gauge conditions $\mathcal{F}_\mu = \frac{1}{\sqrt{\alpha_\text{D}}}\bar{e}_a{}^\nu(\bar{\mathcal{D}}_\nu \varepsilon^a{}_\mu + \beta_\text{D}\bar{\mathcal{D}}_\mu \varepsilon^a{}_\nu)$ and $\mathcal{G}^{ab} = \frac{1}{\sqrt{\alpha_\text{L}}}\bar{g}^{\mu\nu}(\bar{e}^b{}_\nu \varepsilon^a{}_\mu - \bar{e}^a{}_\nu \varepsilon^b{}_\mu)$ respectively. They contain three k-independent gauge parameters $(\alpha_\text{D}, \alpha_\text{L}, \beta_\text{D})$. Using the same notation and conventions as in [19], $\varepsilon^a{}_\mu \equiv e^a{}_\mu - \bar{e}^a{}_\mu$ and $\tau^{ab}{}_\mu \equiv \omega^{ab}{}_\mu - \bar{\omega}^{ab}{}_\mu$ denote the fluctuations of the tetrad and the spin-connection, respectively, and $\bar{\mathcal{D}}_\mu$ is the covariant derivative which contains both the (background) spacetime- and spin-connection.

Our functional RG analysis of selfdual gravity will be carried out using the same Wegner–Houghton-like flow equation and adapted plane wave-based projection techniques as in [19], namely $\partial_t \Gamma_k = \frac{1}{2} D_t \,\text{STr}\Big|_k \ln\big(\Gamma_k^{(2)}\big)$. Here $\text{STr}\Big|_k$ indicated an IR regularization of the supertrace by a sharp cutoff of the momentum integral, and the derivative D_t acts only on the explicit $t \equiv \ln(k)$-dependence due to this cutoff.

In the following we will only highlight the structural differences of the RG equations for selfdual gravity compared to Quantum Einstein–Cartan Gravity (QECG), in subsection 2.1, before we derive its β-functions in subsection 2.2, and proceed with the presentation of the resulting RG flow in subsection 2.3.

2.1. Special features of the selfdual case

Field content. The most obvious modification in comparison to QECG is that we restrict the field space of spin connections to one chirality. Since the projectors $P^\pm = \frac{1}{2}(\mathbb{1} \pm \star)$ decompose any connection according to $\omega = (P^+ + P^-)\omega = \omega^{(+)} + \omega^{(-)}$, this restriction corresponds to halving its number of independent components. Thus we are left with $28 = 16 + 12$ independent field components of vielbein and spin connection, respectively. This is reflected in the dimension of the Hessian $\Gamma_k^{(2)}$ that in the (anti-)selfdual case corresponds to a 28×28-matrix differential operator. We will see in a moment how an adapted decomposition of the fields gives rise to a simple reduction of the 40×40 Hessian of the general Holst truncation to the (anti-)selfdual case.

Gauge symmetry. If we denote the six generators of the full O(4)-gauge group by M_{ab}, with $M_{ab} = -M_{ba}$, by definition they satisfy the algebra

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc})\,. \tag{2.1}$$

A simple computation reveals that the 3 generators $M^\pm_{ab} = (P^\pm M)_{ab}$ of each sign satisfy an algebra of the same form, individually, and that the generators of different $\star$-eigenvalue commute with each other, $[M^\pm_{ab}, M^\mp_{cd}] = 0$. Using the t'Hooft η-symbols [40] that map (anti-)selfdual O(4)-tensors onto SO(3)-vectors it is in fact easy to show that the generators $L^\pm_i = \frac{1}{4}\eta_i{}^{ab}M^\pm_{ab}$ satisfy the usual angular momentum algebra $[L_i, L_j] = i\varepsilon_{ij}{}^k L_k$. Thus the O(4)-algebra decomposes into two chiral factors such that locally also the groups satisfy

$$\text{O}(4) \cong \text{SO}^+(3) \times \text{SO}^-(3). \tag{2.2}$$

This is the Euclidean counterpart of the decomposed Lorentz group SO(3, 1), which is well known to comprise two chiral SU(2) components, too. But in contrast to our case the boost and rotation generators in SO(3, 1) obtain as *complex* linear combinations of the SU(2) components. Moreover, there, the eigenvalues of $\star$ are $\mp i$, whence the (anti-)selfdual components of a real tensor are necessarily complex.

When we restrict ourselves to spin connections of one chirality we, thus, also reduce the gauge group to one chiral component of the above decomposition. In summary, we therefore conclude that the theory space of (anti-)selfdual gravity is reduced in both, the field content and the total symmetry group **G**, and is hence given by the set of action functionals

$$\mathcal{T}^\pm_{\text{EC}} = \Big\{ A[e^a{}_\mu, \omega^{\pm ab}{}_\mu, \cdots] \,\Big|\, \text{inv. under } \mathbf{G} = \text{Diff}(\mathcal{M}) \ltimes \text{SO}^\pm(3)_{\text{loc}} \Big\}\,. \tag{2.3}$$

Here the dots stand for additional background- and ghost-fields.

Gauge conditions and ghost fields. With the reduced gauge group at hand also the 6 gauge fixing conditions $\mathcal{G}_{ab}$ of the former $\text{O}(4)_{\text{loc}}$-group have to be reduced to only 3 that are needed to gauge-fix the remaining $\text{SO}^\pm(3)_{\text{loc}}$ component. Most easily this is done by a projection of $\mathcal{G}_{ab}$ to its (anti-)selfdual part, using now

$$\mathcal{G}^\pm_{ab} = (P^\pm \mathcal{G})_{ab}\,. \tag{2.4}$$

We apply the Faddeev–Popov procedure exactly as before, and find that in S_{gh} simply the O(4)-ghost fields $\bar{\Upsilon}_{ab}$, Υ_{ab} get replaced by their (anti-)selfdual components $\bar{\Upsilon}^\pm_{ab}$, $\Upsilon^\pm_{ab}$. The diffeomorphism gauge-condition $\mathcal{F}_\mu$ gets modified only slightly, since the covariant derivative inside it now is constructed from the (anti-)selfdual spin connection.

Irreducible field parameterization. In order to partially diagonalize the Hessian of the Holst action the fields representing small fluctuations about the background $(\bar{e}, \bar{\omega})$ were parameterized by component fields that transform irreducibly. For the spin connection the corresponding decomposition of $\tau^{ab}{}_\mu \equiv \omega^{ab}{}_\mu - \bar{\omega}^{ab}{}_\mu$ reads