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Boundary terms of conformal anomaly

Sergey N. Solodukhin

Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350, Fédération Denis Poisson, Université François-Rabelais Tours, Parc de Grandmont, 37200 Tours, France

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ABSTRACT

We analyze the structure of the boundary terms in the conformal anomaly integrated over a manifold with boundaries. We suggest that the anomalies of type B, polynomial in the Weyl tensor, are accompanied with the respective boundary terms of the Gibbons–Hawking type. Their form is dictated by the requirement that they produce a variation which compensates the normal derivatives of the metric variation on the boundary in order to have a well-defined variational procedure. This suggestion agrees with recent findings in four dimensions for free fields of various spins. We generalize this consideration to six dimensions and derive explicitly the respective boundary terms. We point out that the integrated conformal anomaly in odd dimensions is non-vanishing due to the boundary terms. These terms are specified in three and five dimensions.

1. Introduction

As is well-known the variational principle for the bulk action which includes functions of the Riemann curvature is not well defined in the presence of boundaries. The variation of the curvature produces a normal derivative of the metric variation on the boundary. The elimination of this term by fixing, additionally to the metric itself on the boundary, also its normal derivative makes the problem over constrained so that no non-trivial solution to the corresponding field equations exists. A way out was found by Gibbons and Hawking [1] in 1977. They suggested that one has to add a boundary term which depends on the extrinsic curvature of the boundary. The role of this term is to cancel the unwanted normal derivatives of the variation of metric. This term for the Einstein–Hilbert action, linear in the curvature, is now known as the Gibbons–Hawking term.

For more general functions which may include polynomials and derivatives of the curvature the appropriate boundary term was found in [2]. In [2] it was used the fact that, by adding auxiliary fields, any function of the curvature can be re-written in a form linear in the Riemann tensor. This allowed to derive a universal form for the boundary term in a very general class of theories.

One of the interesting functionals of the curvature is the integrated conformal anomaly. The local form of the anomaly in four dimensions was established in works of Duff and collaborators, [3]. The general classification of the anomalies made in [4] considers two types of the anomaly. The anomaly of type A is given by the Euler density while the anomaly of type B is constructed from the Weyl tensor $W_{\alpha\beta\mu\nu}$ and its covariant derivatives. In the presence of boundaries one may use the extrinsic curvature of the boundary to construct the conformal invariant quantities. More precisely, it is the traceless part $\hat{K}_{\mu\nu}=K_{\mu\nu}-\frac{1}{d-1}\gamma_{\mu\nu}K$ of the extrinsic curvature ($\gamma_{\mu\nu}$ is the induced metric on the boundary) that transforms homogeneously under conformal transformations, $\hat{K}_{\mu\nu}\to e^{\sigma}\hat{K}_{\mu\nu}$ if $g_{\mu\nu}\to e^{\sigma}g_{\mu\nu}$. Thus, in d dimensions the integrated conformal anomaly may have the following general form

$$\int_{\mathcal{M}_d}\sqrt{g}\langle T_{\mu\nu}\rangle g^{\mu\nu}=a\,\chi(\mathcal{M}_d)+b_k\int_{\mathcal{M}_d}\sqrt{\gamma}I_k(W)$$
$$+b'_k\int_{\partial\mathcal{M}_d}\sqrt{\gamma}J_k(W,\hat{K})+c_n\int_{\partial\mathcal{M}_d}\sqrt{\gamma}\mathcal{K}_n(\hat{K}),\qquad(1)$$

where $\chi(\mathcal{M}_d)$ is the Euler number of manifold $\mathcal{M}_d$, $I_k(W)$ are conformal invariants constructed from the Weyl tensor, $\mathcal{K}_n(\hat{K})$ are polynomials of degree $(d-1)$ of the trace-free extrinsic curvature. In this note we suggest that in some appropriate normalization $b'_k=b_k$ and that the corresponding boundary term $J_k(W,\hat{K})$ is in fact the Hawking–Gibbons type term for the bulk action $I_k(W)$. In dimension $d=4$ this suggestion can be tested by comparing our result with the direct calculation performed recently by Fursaev [5] for free fields of various spins (for scalar fields this was done earlier by Dowker and Schofield [6]). We then extend our

E-mail address: Sergey.Solodukhin@lmpt.univ-tours.fr.

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consideration to dimension $d=6$ and derive the exact form for the boundary terms $J_k(W,\hat{K})$.

Thus, in the presence of boundaries the only new conformal charges which appear to emerge are c_n that are related to the conformal invariant expressions constructed from the trace free extrinsic curvature. The respective terms in the anomaly are interesting since they are present even in flat spacetime. However, as shows the example of scalar field in $d=4$ these charges may depend on the choice of the boundary conditions. Therefore, their invariant meaning is not very clear. It would be interesting to associate these charges with certain structures which appear in the correlation functions of the CFT stress-energy tensor when boundaries are present. Answering this question, however, goes beyond the scope of the present short note.

2. Gibbons–Hawking type boundary terms

In this section we briefly review the construction given in [2] and then adapt it to the invariants constructed from the Weyl tensor. This construction uses the fact that by introducing the auxiliary fields any function of the curvature can be re-written in the form which is linear in the Riemann tensor. In the cases we are interested in this paper it is sufficient to add two auxiliary fields $U_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$. The bulk terms then take the form

$$I_{bulk}=\int_{\mathcal{M}_d}\left(U^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}-U^{\alpha\beta\mu\nu}V_{\alpha\beta\mu\nu}+F(V)\right), \tag{2}$$

where the exact form of $F(V)$ depends on the original form of the action. Then, according to [2] in order to cancel the normal derivatives of the metric variation on the boundary under variation of (2) one should add the corresponding boundary term,

$$I_{boundary}=-\int_{\mathcal{M}_d}U^{\alpha\beta\mu\nu}P^{(0)}_{\alpha\beta\mu\nu}\,,\quad P^{(0)}_{\alpha\beta\mu\nu}=n_\alpha n_\nu K_{\beta\mu}-n_\beta n_\nu K_{\alpha\mu}-n_\alpha n_\mu K_{\beta\nu}+n_\beta n_\mu K_{\alpha\nu}\,. \tag{3}$$

If the bulk invariant is expressed in terms of Weyl tensor only, the above procedure produces the following result for a manifold with boundary

$$I[W]=\int_{\mathcal{M}_d}\left(U^{\alpha\beta\mu\nu}W_{\alpha\beta\mu\nu}-U^{\alpha\beta\mu\nu}V_{\alpha\beta\mu\nu}+F(V)\right)-\int_{\mathcal{M}_d}U^{\alpha\beta\mu\nu}P_{\alpha\beta\mu\nu}\,, \tag{4}$$

where we introduced

$$P_{\alpha\beta\mu\nu}=P^{(0)}_{\alpha\beta\mu\nu}-\frac{1}{d-2}(g_{\alpha\mu}P^{(0)}_{\beta\nu}-g_{\alpha\nu}P^{(0)}_{\beta\mu}-g_{\beta\mu}P^{(0)}_{\alpha\nu}+g_{\beta\nu}P^{(0)}_{\alpha\mu})+\frac{P^{(0)}}{(d-1)(d-2)}(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\nu})\,,$$
$$P^{(0)}_{\mu\nu}=n_\mu n^\alpha K_{\alpha\beta}+n_\mu n^\alpha K_{\alpha\nu}-K_{\mu\nu}-n_\mu n_\nu K\,,\quad P^{(0)}=-2K\,, \tag{5}$$

where we used that $n^\alpha n^\beta K_{\alpha\beta}=0$. $P_{\alpha\beta\mu\nu}$ has same symmetries as the Weyl tensor. In particular, $P^\alpha{}_{\mu\alpha\nu}=0$.

An interesting property of $P_{\alpha\beta\mu\nu}$ is that it does not change if we redefine extrinsic curvature,

$$K_{\mu\nu}\to K_{\mu\nu}-\lambda\gamma_{\mu\nu}\,,\quad P_{\alpha\beta\mu\nu}\to P_{\alpha\beta\mu\nu}\,, \tag{6}$$

where $\gamma_{\mu\nu}=g_{\mu\nu}-n_\mu n_\nu$ is the induced metric on the boundary. Under the conformal transformations, $g_{\mu\nu}\to e^{2\sigma}g_{\mu\nu}$, the extrinsic curvature changes as $K_{\mu\nu}\to e^\sigma(K_{\mu\nu}-\gamma_{\mu\nu}n^\alpha\partial_\alpha\sigma)$. Therefore, the invariance (6) indicates that $P_{\alpha\beta\mu\nu}$ is a conformal tensor which transforms homogeneously under the conformal rescaling of metric, $P_{\alpha\beta\mu\nu}\to e^{3\sigma}P_{\alpha\beta\mu\nu}$. Invariance (6) also means that $P_{\alpha\beta\mu\nu}$ can be rewritten entirely in terms of the trace free extrinsic curvature $\hat{K}_{\mu\nu}=K_{\mu\nu}-\frac{1}{d-1}\gamma_{\mu\nu}K$. The latter is of course consistent with the conformal symmetry of $P_{\alpha\beta\mu\nu}$.

Let us consider some examples.

1. $I[W]=\int_{\mathcal{M}_d}\mathrm{Tr}\,(W^n)$. In this case we have

$$F(V)=\mathrm{Tr}\,(V^n)\,,\quad V=W\,,\quad U=nW^{n-1}\,. \tag{7}$$

After resolving equations for V and U one finds for a manifold with boundary

$$I[W]=\int_{\mathcal{M}_d}\mathrm{Tr}\,(W^n)-\int_{\partial\mathcal{M}_d}n\,\mathrm{Tr}\,(PW^{n-1})\,, \tag{8}$$

where P is defined in (5).

2. $I[W]=\int_{\mathcal{M}_d}\mathrm{Tr}\,(W\Box W)$. In this case we have

$$F(V)=\mathrm{Tr}\,(V\Box V)\,,\quad V=W\,,\quad U=2\Box W \tag{9}$$

and after resolving equations for V and U we find

$$I[W]=\int_{\mathcal{M}_d}\mathrm{Tr}\,(W\Box W)-2\int_{\partial\mathcal{M}_d}\mathrm{Tr}\,(P\Box W)\,. \tag{10}$$

These examples will be useful in the subsequent sections.

3. Conformal anomaly in $d=4$

In four dimensions the local form of the anomaly is well-known

$$\langle T\rangle=-\frac{a}{5760\pi^2}E_4+\frac{b}{1920\pi^2}\mathrm{Tr}\,W^2\,,$$
$$E_4=R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-4R_{\mu\nu}R^{\mu\nu}+R^2\,,$$
$$\mathrm{Tr}\,W^2=R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}R^2\,. \tag{11}$$

In this normalization a scalar field has $a=b=1$. The integrated conformal anomaly contains the bulk integrals of the rhs of (11) and some boundary terms. In particular, the bulk integral of E_4 is supplemented by some boundary terms to form a topological invariant, the Euler number,

$$\chi[\mathcal{M}_4]=\frac{1}{32\pi^2}\int_{\mathcal{M}_4}E_4-\frac{1}{4\pi^2}\int_{\partial\mathcal{M}_4}(K^{\mu\nu}R_{n\mu n\nu}-K^{\mu\nu}R_{\mu\nu}-KR_{nn}+\frac{1}{2}KR-\frac{1}{3}K^3+K\,\mathrm{Tr}\,K^2+\frac{2}{3}\mathrm{Tr}\,K^3)\,, \tag{12}$$

where $R_{\mu n\nu n}=R_{\mu\alpha\nu\beta}n^\alpha n^\beta$ and $R_{nn}=R_{\mu\nu}n^\mu n^\nu$. This form for the boundary terms was found in [6].

On the other hand, the integral of the Weyl tensor squared should be supplemented by a boundary term as we explained in the previous section, see eq. (8) for $n=2$,

$$\int_{\mathcal{M}_4}\mathrm{Tr}\,W^2-2\int_{\partial\mathcal{M}_4}\mathrm{Tr}\,(WP)\,. \tag{13}$$

The properties of the Weyl tensor insure a simplification: